A Reynolds' Limit Formula for the Shear Stress in Dorodnitzyn's Boundary Layer

Carla V. Valencia-Negrete*

* *Department of Physics and Mathematics, Universidad Iberoamericana, A. C., Mexico City, Mexico*

Summary. Shear Stress growth is an indicator of Boundary Layer separation. The main difficulty to obtain clear descriptions of its behavior lies in Navier-Stokes Equations' non-linearity. On the other hand, Dorodnitzyn stated a Gaseous Boundary Layer problem, valid in atmospheric conditions, and deduced a second-order quasi-linear problem for a transformation of the Shear Stress. This article presents a mathematical formalization of this last problem and a Reynolds' Limit Formula for it, deduced with Bayada and Chambat's change of variables. For general compressible Reynolds' equations, the problem was solved by Chupin and Sart in 2012. Undoubtedly, there is a mathematical formalization for Dorodnitzyn's model previous from the one that is given here, but the author has not been able to find it in the literature. In earlier work, the author verified his first step simplification. Now, the formalization is extended to Dorodnitzyn's second-order quasi-linear problem. Then, the small parameter problem is deduced, and a bound, independent of the parameter, is found in the corresponding Sobolev Space to prove the existence of a Reynolds' Limit Formula for Dorodnitzyn's Shear Stress problem.

Abstract

The Earth's Global Mean Temperature is going to increase by, at least, $1.5°$C in the next 10 to 33 years [3, p. 6]. As a consequence, there will be an increment in the number of severe droughts and flooding [3, p. 9]. Its origin, atmospheric convection, could be studied as a boundary layer separation problem. To identify its sources and sinks, the suggestion is to study *shear stress* growth deduced from approximate gaseous boundary layer models in atmospheric conditions.

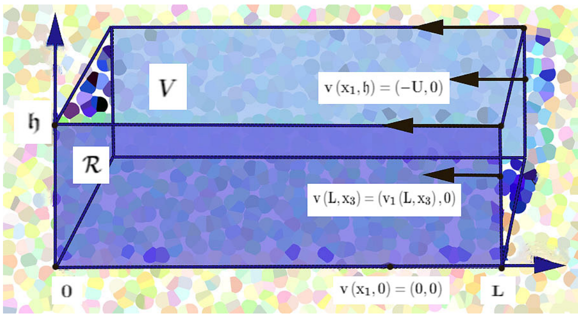

Figure 1: Dorodnitzyn's Rectangular Domain $\mathcal{R}$= $(0, L) \times (0, h) \in \mathbb{R}^2$

In 1942, Dorodnityzn stated a Gaseous Boundary Layer problem [2] in a rectangle $R = (0, L) \times (0, h) \subset \mathbb{R}^2$, where $L >> h > 0$ [2], of three simplified stationary Conservation of Mass, Conservation of Momentum, and Conservation of Energy laws, Eq. (1), (2) and (3),

$$\frac{\partial\,(\rho\,u)}{\partial x} + \frac{\partial\,(\rho\,v)}{\partial y} \;=\; 0\,; \tag{1}$$

$$\rho\left(u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y}\right) \;=\; -\,\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu\,\frac{\partial u}{\partial y}\right); \quad \text{y} \tag{2}$$

$$\rho\left[u\,\frac{\partial\,(c_p\,T)}{\partial x} + v\,\frac{\partial\,(c_p\,T)}{\partial y}\right] \;=\; \frac{\partial}{\partial y}\left[\kappa\,\frac{\partial T}{\partial y}\right] + \mu\,\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial p}{\partial t}, \tag{3}$$

where one can assume that the *stationary density* $\rho \in L^2\,(R; (0, \infty))$; that the horizontal *velocity component* $u \in L^2\,(R)$ has *generalized derivatives* $\partial u/\partial x,\ \partial u/\partial y,\ \partial^2 u/\partial y^2 \in L^2\,(R)$; the vertical *velocity component* $v \in L^2\,(R)$; the *absolute temperature* $T \in L^2\,(R; (0, \infty))$ with $\partial T/\partial y,\ \partial^2 T/\partial y^2 \in L^2\,(R)$; the *dynamic viscosity* $\mu \in L^2\,(R)$, the *pressure* $p \in L^2\,(R)$, the *thermal conductivity* $\kappa \in L^2\,(R)$, all of them with first order generalized derivatives in $L^2\,(R)$; and both products $\rho\,u,\ \rho\,v \in L^2\,(R)$. This is, assume ρ, u, v, T, μ, p and κ are elements of the space $W^{1,2}\,(R)$, so that a *Leibnitz Rule* for product differentiation is valid in the non-empty open domain $R \subset \mathbb{R}^2$ when both factors and all the *generalized derivatives* involved are elements of $L^2(R)$ [4, p. 11]. The value c_p is the *specific heat at constant pressure for dry air*, and we have four Ideal Gas Thermodynamic Laws, Eq. (4), (5), (6), (7): the *Prandtl number* $Pr = 1$,

$$Pr \;=\; \frac{c_p\,\mu}{\kappa} = 1; \tag{4}$$

the *Equation of State* for the *Universal Gas Constant* R^*, a volume $V = \iiint_B dx\;dy\;dz$ of a ball $B(r, \mathbf{x_0}) \subset \mathbb{R}^3$ of positive radius $r > 0$ and center $\mathbf{x_0} = (x_0, y_0, z_0)$ such that $(x_0, y_0) \in R$ and $R \times \{0\} \subset B$, and the number of moles n of an ideal gas corresponding to the volume V,

$$p\,V \;=\; n\,R^*\,T; \tag{5}$$

the *adiabatic polytropic atmosphere* [7, p. 35] where $b = 1.405$ and c are constants,

$$p\,V^b \;=\; c; \tag{6}$$

and the *Power Law* [6, p. 46]

$$\frac{\mu}{\mu_h} \;=\; \left(\frac{T}{T_h}\right)^{\frac{19}{25}}. \tag{7}$$

The boundary conditions, Eq. (8), (9), (10), (11), (12), (13), are given by the *free-stream velocity* $U > 0$, the *no slip condition*,

$$(u,v)|_{\{(x,h):\,0\le x\le L\}} \;=\; (-U,0), \tag{8}$$

$$(u,v)|_{\{(x,0):\,0\le x\le L\}} \;=\; (0,0), \tag{9}$$

the *free-stream temperature* $T_h > 0$, the *free-stream dynamic viscosity* $\mu_h > 0$,

$$T|_{\{(x,h):\,0\le x\le L\}} \;=\; T_h > 0, \tag{10}$$

$$\mu|_{\{(x,h):\,0\le x\le L\}} \;=\; \mu_h > 0. \tag{11}$$

periodic conditions for all $y \in [0,h]$:

$$(u\,(0,y)\,,0) \;=\; (u\,(L,y)\,,0)\,; \tag{12}$$

and a Neumann condition:

$$\left.\frac{\partial T}{\partial y}\right|_{\{(x,0):\,0\le x\le L\}} \;=\; 0. \tag{13}$$

As Busemann previously did in 1935, Dorodnitzyn expressed T in terms of u, but considers the Conservation of Energy Law in terms of the *total energy per unit mass*, $E = c_pT + u^2/2$, in the form presented by Luigi Crocco in 1932. Additionally, he includes a pressure variation term, $\partial p/\partial x$, and so allows the possibility of a Boundary Layer separation. By a successive substitution of $T(u)$, this system of seven equations is reduced to a system of just two with inherited boundary conditions in terms of a *stream function* defined, in the formalisation, by means of a generalized *Green's Theorem* [5, p. 121] that is valid for elements of the Sobolev Spaces $W^{1,2}(R)$. Moreover, he defined a *diffeomorphism* $R \xrightarrow{\mathbf{s}} \Pi$ that allows writing von Kármán's Integral Formula for a compressible fluid in an incompressible form in a polygonal domain $\Pi = \mathbf{s}(R)$ where $(x,y) \xmapsto{\mathbf{s}} (\ell,s)$, $\ell\,(\hat{x},\hat{y}) \longmapsto \int_0^{\hat{x}} p\,(x,\hat{y})\;dx$ and $s\,(\hat{x},\hat{y}) \longmapsto \int_0^{\hat{y}} \rho\,(\hat{x},y)\;dy$. This way, he opens the road to adapt Blasius' method to state the *stream function* problem as an Ordinary Differential Equation.

In order to do this, he applies a subsequent *diffeomorphism* $\Pi \xrightarrow{\mathbf{z}} S$ that takes the polygon Π into a strip band of infinite positive heights $S = \mathbf{z}(\Pi)$ with $(\ell,s) \xmapsto{\mathbf{z}} (\ell,z)$ and $z\,(\ell,s) \longmapsto s/\sqrt{\ell}$. In terms of z, the *shear stress* $\tau = \mu\;\partial u/\partial y$ becomes $\tau_s(z) = (a\;x^{1/2}\;\tau) \circ \mathbf{s}^{-1} \circ \mathbf{z}^{-1}(z)$ for a constant a. If we denote $u_s(z) = u \circ \mathbf{s}^{-1} \circ \mathbf{z}^{-1}(z)$, $i_0 = c_p\;T_0$ and $\sigma_0 = 1 - \left(U^2/2i_0\right)$ where $T_0 = T_h + U^2/(2c_p)$ is the absolute temperature value at height $y = 0$ in R, then $(u,v,T,p,\rho,\mu,\kappa)$ is a *classical solution* of Eq. (1), (2), (3), (4), (5), (6), (7) with boundary conditions (8), (9), (10), (11), (12), (13) if and only if $\tau_s \in C^1(0,\infty)$ satisfies the second-order quasi-linear problem:

$$\tau_s\;\frac{\partial^2\tau_s}{\partial u_s^2} = -\,A\;u_s\;\left(1-\frac{u_s^2}{2i_0}\right)^{-6/25}, \tag{14}$$

with inherited boundary conditions and $A = 1/2\cdot(n\;R^*\;T_0)/V\cdot\;T_0^{\frac{2b}{b-1}}\cdot\sigma_0^{1-\frac{b}{(b-1)}}$. Bayada and Chambat's change of variables $R \xrightarrow{\phi^\epsilon} R^\epsilon$ for $\epsilon = h/L > 0$ with $(x,y) \xmapsto{\phi^\epsilon} (x/L, y/\,(L\epsilon))$ provides a small parameter problem [8] for the sequence $(\mathbf{v}^\epsilon)_\epsilon = (u^\epsilon, v^\epsilon)_\epsilon$ where $u^\epsilon = \frac{1}{L}\;u$ and $v^\epsilon = \frac{1}{L\epsilon}\;v$. This way, there is a inherent adimensional problem for the sequence (τ_s^ϵ) so that the existing bound found for (u^ϵ) in [8] is valid for (τ^ϵ), and we can derive a Reynolds' Limit Formula for Dorodnitzyn's Shear Stress problem.

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