Non-Conservative Nonlinear Modes Through Energy Resonance

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Summary. In this contribution, a method for the numerical computation of nonlinear modes for damped systems is presented. The main idea is to introduce a conservative force term to compensate the non-conservative effects, in such a way that the condition of energy resonance is verified. We describe a systematic approach for the construction of the associated backbone curves and discuss related subjects such as their stability and practical applications.

Introduction

Nonlinear Normal Modes (NNM), which extend the familiar modal analysis to the case of nonlinear mechanical systems, have been a central research topic in dynamics over the past decades. Besides being an interesting theoretical concept, they have been shown to lead to numerous practical applications thanks to their relation to forced resonances, localization, energy transfers and model reduction (see [1] for a review, as well as references therein). Simply stated, the idea behind all methods for the computation of NNMs is to construct an invariant manifold of the conservative equations of motion, either directly or through the continuation of periodic solutions on the manifold [2]. However, modes computed in this fashion do not take into account the presence of non-conservative effects. As a consequence, deviations from the actual backbone curves of the system are to be expected: while they remain small in the case of small proportional damping, they can be considerable when damping is large or when the nonlinear forces involved are fundamentally non-conservative. Hence, just as for the conservative case, efforts have been made to extend the concept of non-conservative modes to non-linear systems, yielding Non-Conservative Nonlinear Modes (NCNM). Different methodologies for the computation of NCNMs exist, relying on generalizations of the manifold-construction, complex-mode and periodic-motion-continuation techniques. The latter of these, introduced by Krack [3], consists in adding an artificial damping term to compensate for all non-conservative effects and thus achieve periodic motions. This method was applied by Jahn and co-workers [4] to study self-excited vibrations, and in particular to detect limit cycle oscillations. Nevertheless, one of the assumptions of this method is that the non-conservative terms are frequency-independent, which is not the case of, e.g., systems with memory terms. The present contribution introduces a novel, general method for the computation of NCNMs, based on extending the concept of energy resonance from linear modal analysis to the nonlinear case. Upon application of the Harmonic Balance Method (HBM), this leads to the formulation of a conservative equation where the effect of non-conservative terms of any form can be accurately taken into account, and whose solutions correspond to NCNMs.

Energy Resonance

Consider the equations of motion for a system of damped, unforced, nonlinear oscillators. By applying the HBM, they are expressed in the frequency domain with the Fourier coefficients of displacements $\mathbf{X}$ and the fundamental frequency ω as unknowns, giving:

$$\mathbf{R}(\mathbf{X},\omega) = \mathbf{Z}(\omega)\mathbf{X} + \mathbf{F}_{\mathrm{NL}}(\mathbf{X},\omega) = \mathbf{0} \tag{1}$$

where $\mathbf{Z}(\omega) = \omega^2\nabla^2 \otimes \mathbf{M} + \omega\nabla \otimes \mathbf{C} + \mathbf{I} \otimes \mathbf{K}$ is the dynamical stiffness matrix containing inertial, damping and stiffness terms, and $\mathbf{F}_{\mathrm{NL}}(\mathbf{X},\omega)$ represents the Fourier coefficients of all nonlinear forces. Excluding the case of limit cycle oscillations, this equation has no non-trivial solution since it contains non-conservative terms and thus no periodic solutions. If these terms are ignored, a conservative system is obtained, whose solution yields the traditional, undamped nonlinear modes as per Rosenberg's definition:

$$\mathbf{R}_{\mathrm{P}}(\mathbf{X},\omega) = \left[\omega^2\nabla^2 \otimes \mathbf{M} + \mathbf{I} \otimes \mathbf{K}\right]\mathbf{X} + \mathbf{F}_{\mathrm{NL},c}(\mathbf{X},\omega) = \mathbf{0} \tag{2}$$

Drawing a parallel with linear modal analysis, these solutions describe phase resonance. Another possibility, as proposed by Grenat et al. [5], is to compensate the non-conservative terms through the addition of a fictitious conservative force in such a way that the underlying invariant manifold is kept unchanged. This is achieved by considering the condition for energy -rather than phase- resonance, which can be expressed by the scalar equation:

$$\frac{\partial\left(\mathbf{X}^T\mathbf{X}\right)}{\partial\omega} = 0 \tag{3}$$

In this work, we show that Eq. (3) leads to the following conservative system:

$$\mathbf{R}_{\mathrm{E}}(\mathbf{X},\omega) = \left(\omega^2\nabla^2 \otimes \mathbf{M} + \mathbf{I} \otimes \mathbf{K}\right)\mathbf{X} + \mathbf{F}_{\mathrm{NL},c}(\mathbf{X},\omega) - \mathbf{G}_{\mathrm{D}}(\mathbf{X},\omega) = \mathbf{0} \tag{4}$$

where the non-linear terms have been split into a conservative ($\mathbf{F}_{\mathrm{NL},c}$) and a non-conservative part ($\mathbf{F}_{\mathrm{NL},nc}$). The term $\mathbf{G}_{\mathrm{D}}$, for which we have derived an explicit analytical expression, is a function of all non-conservative terms, both linear and nonlinear. Eq. (4) can be used to compute normal mode solutions regardless of the specific form of the non-conservative terms, under the condition that they are continuous functions of $\mathbf{X}$ and that, as for any autonomous system, a phase equation be appended for closure. Illustrative examples are presented below:

1. Viscous damping
Assuming linear damping of the form: $\mathbf{f}_{\text{nc}}(t) = \mathbf{C}\dot{\mathbf{x}}(t)$:

$$\mathbf{G}_{\text{D}} = \frac{1}{2}\mathbf{I} \otimes \left(\mathbf{M}^{-1}\mathbf{C}\mathbf{C}\right)\mathbf{X} \tag{5}$$

Fig. 1 a) superimposes the frequency response of a Duffing oscillator with high damping to its corresponding NNM and NCNM backbones, for varying excitation levels.

2. Flow-induced added damping
Following [7], forces induced by transverse flow on a flexible cylinder within a rigid array can be modelled by a memory function of the form: $\mathbf{f}_{\text{fe}}(\mathbf{x}, t) = \sum_{k=0}^{N} \mathbf{A}_k \int_0^t e^{-b_k \tau}\mathbf{x}(t-\tau)d\tau$. This leads to added damping terms which are linear in $\mathbf{X}$ but frequency-dependent through the matrices $\mathbf{D}_k(\omega, b_k)$:

$$\mathbf{G}_{\text{D}} = \frac{1}{2}\left\{\sum_{k=0}^{N}\left[\mathbf{I}\otimes\mathbf{M} + b_k\mathbf{D}_k\otimes\mathbf{A}_k\right]^{-1}\left[\mathbf{I}\otimes\mathbf{C} - (\mathbf{D}_k + \omega\nabla^2\mathbf{D}_k^2)\otimes\mathbf{A}_k\right]\left[\mathbf{I}\otimes\mathbf{C} - \mathbf{D}_k\otimes\mathbf{A}_k\right]\right\}\mathbf{X} \tag{6}$$

3. Doubly-clamped viscoelastic beam
Considering a Kelvin-Voigt model [6] to describe the stress-strain relation within the beam, the equations of motion for modal coordinates q_j include non-conservative terms of the form: $\eta q_i q_j \dot{q}_k$ as a result of mid-plane stretching. Fig.1 b) shows the NCNM curves near a 1:5 internal resonance between the first and third bending modes, where considerable deviation from the undamped case is observed for high values of η.

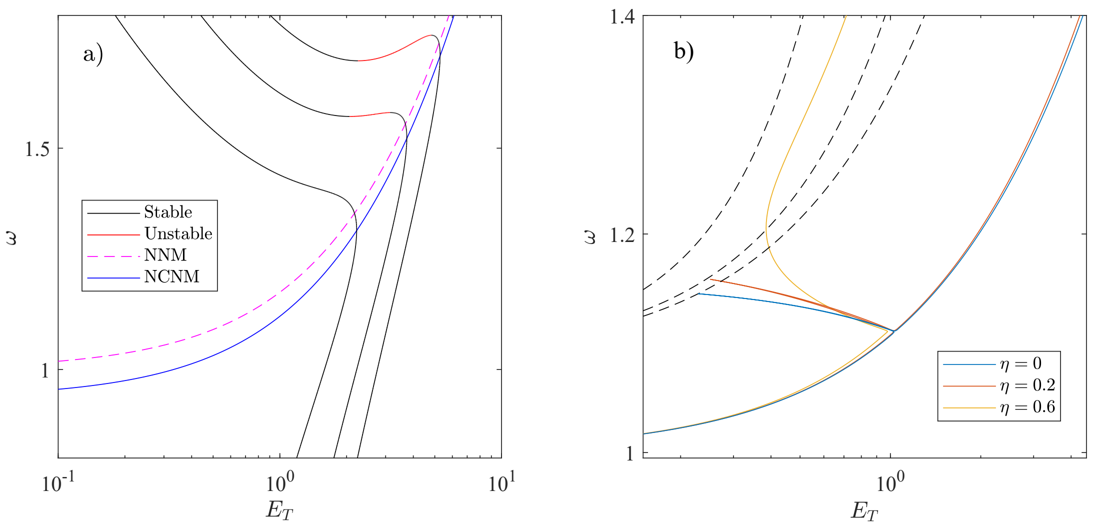

Figure 1: Examples of backbone curves with NCNM formalism. (a) Internal resonance of viscoelastic beam: first (solid) and third (dashed) nonlinear modes for different values of η. (b) Duffing equation with high viscous damping, contrasting NNM and NCNM backbone curves.

Conclusions

In this contribution, we introduced a systematic method for NCNM construction based on the concept of energy resonance. Likewise, its application to diverse mechanical vibration problems was showcased. Further work on this subject includes: bifurcation analysis, a detailed comparison with alternative methods and the experimental targeting of particular nonlinear modes by exploiting the function $\mathbf{G}_{\text{D}}$.

References

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