The Basin Stability of a Bi-Stable Frictional Oscillator

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Summary. The stability of steady-state solutions is typically assessed by means of local criteria. As an example, the eigenvalues' real parts indicate linear stability of an equilibrium position against small perturbations. However, realistic dynamical systems may exhibit multiple co-existing solutions. If the perturbations are not small, local stability measures have only a limited value for evaluating the effective probability of arriving at one of the multiple solutions. Therefore, global methods are required to evaluate the attractiveness of a state for finite perturbations. This work illustrates the concept of basin stability for a frictional oscillator that exhibits bi-stability. The results indicate how the global basin stability measure can complement conventional stability considerations.

Introduction

Dynamical systems are well-known to exhibit multistability, i.e. multiple stable states co-exist at a given system configuration. In this scenario, only the initial conditions and instantaneous perturbations dictate on which state the system will end up. For example, Gräbner et al. [1] report experimental observations of bi-stability in brake system vibrations. Recently, Jahn et al. [2] have discussed a friction-excited system that exhibits multistability via periodic orbits that compete with chaotic dynamics. Classical linearization-based approaches, such as eigenvalues or Lyapunov exponents, asses a state's stability by investigating the local behavior of small perturbations. The absolute values of those metrics quantify the rate of convergence (for stable states), or divergence (for unstable states), and classify the stability in a binary fashion. For example, the eigenvalue's real part indicates the growth of small perturbations in the vicinity of a fixed point, and hence the equilibrium is stable for a negative real part. However, local linear stability concepts cannot indicate the most desirable state amongst the multiple co-existing ones. This scenario is particularly unsatisfactory if some of the stable states are undesired for reasons of increased vibrations, noise, wear, or fatigue. The concept of basin stability [3] relates to the volume of the basin of attraction $\mathcal{B}$. The basin stability $S_{\mathcal{B}}$ of a particular state indicates the volumetric fraction of initial states that converge to the attracting set with respect to the overall state space volume. Therefore, it is a global nonlinear measure for the attractiveness of states, i.e. the basin stability can adequately assess the stability of a state under non-small perturbations. In practical system dynamics, the knowledge of multiple solutions and their stability, as typically displayed in bifurcation diagrams, is only one piece of information. The likelihood of the system arriving at one of those solutions may be of even greater importance during the operation of a mechanical structure or machine. This works strives to illustrate the concept of basin stability on the example of a bi-stable frictional oscillator.

Methods

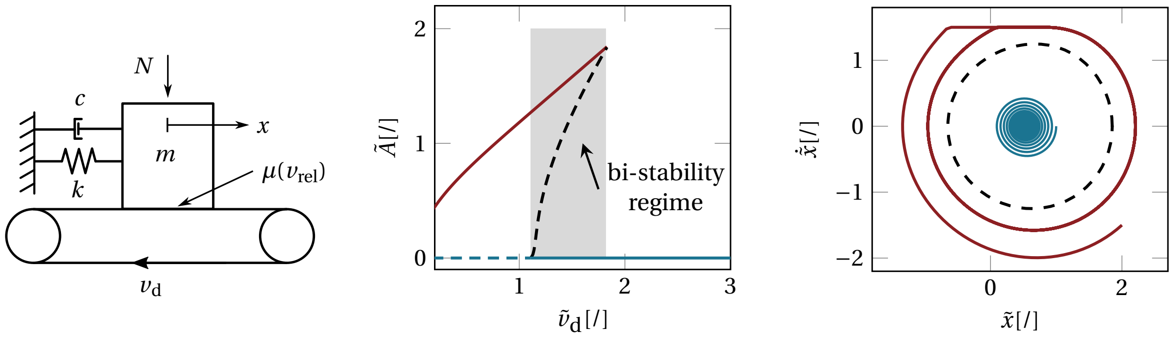
We study the dynamics and stability of a single-degree-of-freedom system $m\ddot{x}+c\dot{x}+kx = F$, see Figure 1 (a), following

$$\begin{aligned} F &= -N\mu\left(v_{\mathrm{rel}}\right)\operatorname{sign}\left(v_{\mathrm{rel}}\right), \quad && v_{\mathrm{rel}} \neq 0, \quad v_{\mathrm{rel}} = \dot{x} - v_{\mathrm{d}} \\ |F| &< \mu_{\mathrm{st}} N, && v_{\mathrm{rel}} = 0, \\ \mu(v_{\mathrm{rel}}) &= \mu_{\mathrm{d}} + (\mu_{\mathrm{st}} - \mu_{\mathrm{d}}) \exp\left(-\frac{|v_{\mathrm{rel}}|}{v_0}\right), \end{aligned} \tag{1}$$

that experiences friction-induced vibrations (FIV). The friction formulation features a velocity-dependent weakening-strengthening behavior that gives rise to an instability of the steady sliding solution, see Figure 1 (b). A self-excited stick-slip limit cycle exists for the parameter range $0 \leq \tilde{v}_{\mathrm{d}} \leq 1.84$ and, most importantly, co-exists with the steady sliding solution in the bi-stability regime $1.11 \leq \tilde{v}_{\mathrm{d}} \leq 1.84$ resulting from the subcritical Hopf bifurcation. Hence, in this parameter range, there exist two stable states in parallel, as depicted in Figure 1 (c). Depending on the initial condition or instantaneous perturbations, the system will either end up in the low-energy steady sliding state, or on the high-intensity stick-slip cycle.

Results

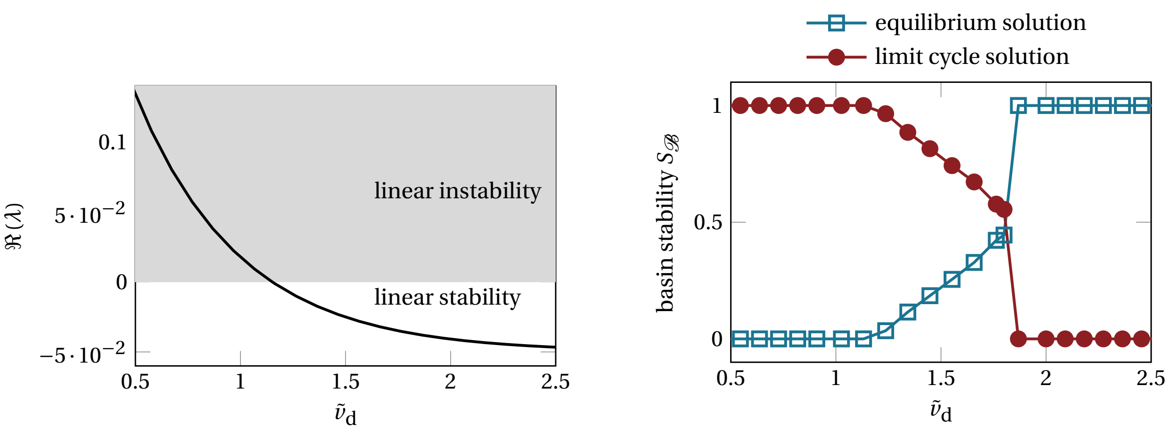
Conventionally, the stability of the equilibrium solution, i.e. the steady sliding state, is assessed by the eigenvalue's real part $\Re\left(\lambda\right)$. However, in realistic systems, it is often unknown what type of perturbations the system may experience during operation. In our model, the unstable periodic orbit (UPO) represents the separatrix between the basins of attraction $\lfloor$ for the two stable states. A perturbation of $\tilde{x} < -0.65$ at $\tilde{v}_{\mathrm{d}} = 1.5$ would result in a jump from the steady sliding state to the limit cycle. The eigenvalue in Figure 2 (a) grows for a decline in belt velocity, indicating that the steady sliding state becomes less stable against perturbations up to $\tilde{v}_{\mathrm{d}} = 1.11$. However, the linear stability analysis is unable to show how fast the basin $\mathcal{B}$ of the steady sliding state shrinks for declining belt velocities $\tilde{v}_{\mathrm{d}}$. Hence, the eigenvalue analysis does not detect the critical transition at $\tilde{v}_{\mathrm{d}} = 1.84$ where the systems transits into the bi-stability regime. On the contrary, the basin



(a) **(b)** **(c)**

Figure 1: (a) single-degree-of-freedom frictional oscillator, (b) bifurcation diagram for the non-dimensional belt velocity $\tilde{v}_{\mathrm{d}}$, and (c) phase plane for $\tilde{v}_{\mathrm{d}} = 1.5$. Stable (unstable) solutions are indicated by solid (dashed)lines. The stable steady sliding state (blue spiral trajectory) co-exists with the unstable periodic orbit (black dashed line) and the stable stick-slip limit cycle (red trajectory). The non-dimensional system $(\tilde{\cdot})$ is evaluated for $\mu_{\mathrm{d}} = 0.5$, $\mu_{\mathrm{st}} = 1$, $\xi = 0.05$, $N = 1$ and $\tilde{v}_0 = 0.5$ following the work [4]

stability $S_{\mathcal{B}}$ in Figure 2 (b) indicates the *global degree of stability*, which changes quickly at the transition point. Here, $S_{\mathcal{B}}$ is a much more reliable proxy for detecting the critical transition of the steady sliding state. While the linear analysis states a negative eigenvalue at $\tilde{v}_{\mathrm{d}} = 1.5$, $S_{\mathcal{B}}$ indicates that the likelihood of converging to the steady sliding state is in fact only $S_{\mathrm{B}} = 0.255$. Hence, the system is three times more likely to exhibit stick-slip vibrations if the initial conditions or perturbations were randomly drawn from the given state space regime.



(a) **(b)**

Figure 2: (a) eigenvalue's real part as a function of the belt velocity. The linearly unstable regime for $\Re(\lambda) > 0$, shaded in gray, is reached at $\tilde{v}_{\mathrm{d}} = 1.11$. (b) basin stability $S_{\mathcal{B}}$ of the steady sliding equilibrium and the stick-slip limit cycle. Initial conditions were drawn from $\tilde{x}_0 \in \{-2, 3\}$ and $\dot{\tilde{x}}_0 \in \{-2, 2\}$ using a uniform grid of 20×20 points.

Conclusions

The basin stability is studied as a measure for the state's relevance in a multistability scenario. While the (nonlinear) stability of this small oscillator is rather straight-forward, the basin stability seems to be a particularly useful stability measure for more complex, i.e. higher-dimensional, systems that feature a complicated multi-stability behavior [5]. Regarding the actual motion during operation, the basin stability is likely to contribute to higher prediction quality of numerical models of mechanical structures. As a second model, a frictional oscillator with multiple degrees of freedom and a more complicated dynamical behavior [2] will be studied in the full conference proceeding.

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