Multiple scale expansion and frequency-response curves of a nonlinear beam model

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Summary. Slender and highly flexible structures quite often take place in systems designed to meet high to extreme performances. Hence cables, ropes, yarns, hoses and pipelines, which are essential parts of such structures, play a relevant role in practically every engineering field. In mechanical and automotive engineering, large amplitude motions of thin rods can be exploited to design nonlinear vibration absorbers for the reduction of torsional vibrations of drivelines; in assembly and disassembly phases and in system operation, reliable models are needed to predict and analyze the behavior of cables and wiring harnesses, taking also into account effective material properties; accurate structural models of wire ropes are required to study the behavior of rope-ways and cranes on the system level. In aerospace engineering, compact, flexible and slender aerials and booms to be deployed in space are typically used to minimize the room needed to store satellites in launching phases. In textile engineering, complicate interactions among hundreds of yarns have to be controlled to obtain the desired final layout. In biomedical engineering, medical endoscopes characterized by a multilayer structure must be accurately modeled, since they exhibit highly deformed configurations while moving inside narrow curved tubes within the human body. In offshore engineering, floaters, mooring lines, and others structural components of floating wind farms, are subject to structural fatigue and various sources of damping and power cables show complex cross-sectional properties. In civil engineering, estimates of the structural properties from response data coming from non-destructive procedures is critically empowered by a deeper understanding of beam-like structures. However, despite of their ubiquity, slender structures in real operating conditions exhibit responses often too complicated for current modeling tools. In this respect there is a continuous need for reliable models. In this area, this contribution considers a beam model equipped with non-standard constitutive laws and in particular it is aimed at deriving approximate solutions of the equations of motion via asymptotic multiple scale expansion.

Introduction

We consider a geometrically exact beam model deduced by stipulating a relation between one- and three-dimensional formulations for initially straight beams undergoing planar and twist-less deformed states. Using comma notation for derivatives, the equations of motion derived in [1], to which we refer for any further detail, are written as

$$m_0\, u_{,tt} - m_1(\theta_{,t}^2 \sin\theta - \theta_{,tt}\cos\theta) + c_0\, u_{,t} + c_1\theta_{,t}\cos\theta = \left(N(1+u_{,x})\sqrt{2\varepsilon+1} - \frac{T v_{,x}}{\sqrt{2\varepsilon+1}}\right)_{,x} + q_1\,, \tag{1}$$

$$m_0\, v_{,tt} - m_1(\theta_{,t}^2 \cos\theta + \theta_{,tt}\sin\theta) + c_0\, v_{,t} - c_1\theta_{,t}\sin\theta = \left(N v_{,x}\sqrt{2\varepsilon+1} + \frac{T(1+u_{,x})}{\sqrt{2\varepsilon+1}}\right)_{,x} + q_2\,, \tag{2}$$

$$m_1(u_{,tt}\cos\theta - v_{,tt}\sin\theta) + m_2\,\theta_{,tt} + c_1(u_{,t}\cos\theta - v_{,t}\sin\theta) + c_2\,\theta_{,t} = M_{,x} - T\sqrt{2\varepsilon+1} + q_3\,, \tag{3}$$

where $u(x,t)$ and $v(x,t)$ stand for the axial and transverse displacements of the beam axis, $\theta(x,t)$ is the cross-sectional rotation, $N(x,t)$, $T(x,t)$ and $M(x,t)$ are axial, transverse and bending generalized stresses. These are related to the axial strain ε, the shear angle γ and the Lagrangian bending curvature κ, all nonlinear functions of u, v, and θ, by the nonstandard constitutive assumptions

$$N = \frac{\sin^2\gamma}{\sqrt{2\varepsilon+1}}\,K_S + \frac{\varepsilon}{\sqrt{2\varepsilon+1}}\,K_0 + \cos\gamma\left(\frac{3\varepsilon+1}{2\varepsilon+1}\right)\kappa\,K_1 + \frac{1+2\cos^2\gamma}{2\sqrt{2\varepsilon+1}}\kappa^2\,K_2 + \frac{\cos\gamma}{2(2\varepsilon+1)}\kappa^3\,K_3\,, \tag{4}$$

$$T = \sqrt{2\varepsilon+1}\,\frac{\sin 2\gamma}{2}\,K_S - \varepsilon\sin\gamma\,\kappa\,K_1 - \sqrt{2\varepsilon+1}\,\frac{\sin 2\gamma}{2}\,\kappa^2\,K_2 - \frac{\sin\gamma}{2}\kappa^3\,K_3\,, \tag{5}$$

$$M = \varepsilon\sqrt{2\varepsilon+1}\,\cos\gamma\,K_1 + \left(\varepsilon + (2\varepsilon+1)\cos^2\gamma\right)\kappa\,K_2 + \frac{3}{2}\sqrt{2\varepsilon+1}\,\cos\gamma\,\kappa^2\,K_3 + \frac{1}{2}\kappa^3\,K_4\,. \tag{6}$$

In Eqs. (1-6), the mass, damping and stiffness coefficients are given by

$$m_i = \int_{\mathcal{S}_0} \rho y^i \mathrm{d}A\,, \quad c_i = \int_{\mathcal{S}_0} c y^i \mathrm{d}A\,, \quad K_i = \int_{\mathcal{S}_0} \mathcal{E}\, y^i\,\mathrm{d}A\,, \quad K_S = \int_{\mathcal{S}_0} \mathcal{G}\mathrm{d}A\,, \tag{7}$$

where ρ, c, $\mathcal{E}$ and $\mathcal{G}$ are mechanical parameters and $\mathcal{S}_0$ is the rigid cross section.
Since their introduction, Eqs. (1-3) have been analyzed in some depth [2, 3, 4], mainly through numerical investigations. On the contrary, the present paper, following [5, 6], is focused on analytical developments, based on the method of multiple scales [7]. In particular, to draw the frequency-response curves, the exact partial differential Eqs. (1-3) are analyzed around frequencies corresponding to certain natural bending modes.

Some preliminaries on multiple time scale equations of motion

We introduce three time scales and develop the time and time derivative operator respectively as $t = \tau_0 + \epsilon\tau_1 + \epsilon^2\tau_2$ and $(\cdot)_{,t} = (\cdot)_{,\tau_0} + \epsilon(\cdot)_{,\tau_1} + \epsilon^2(\cdot)_{,\tau_2}$, being $\epsilon \ll 1$ a book-keeping parameter, and assume that the unknowns u, v, and θ in Eqs. (1-3) are small of order ϵ at most and can be expanded, up to the 3rd order, as

$$w_i(x,t) = \epsilon W_{i1}(x;\tau_0,\tau_1,\tau_2) + \epsilon^2 W_{i2}(x;\tau_0,\tau_1,\tau_2) + \epsilon^3 W_{i3}(x;\tau_0,\tau_1,\tau_2)\,, \qquad i = 1,2,3 \tag{8}$$

where w_i are dummy functions such that $u = w_1$, $v = w_2$, $\theta = w_3$, and W_{ij} are unknown functions to be determined.
Based on appropriate choices of the geometric or mechanical properties of the beam, the loads, or even the reference frame, we can accept that some mechanical parameters are zero or negligible for some power of ϵ. In what follows, we assume that c_0 and K_3 are at most of order ϵ^2; m_1, c_1, c_2, K_1 and q_2 are at most ϵ^3; q_1 and q_3 are at most ϵ^4.
We also assume that the loads are periodic functions as

$$q_i(t) = P_i \cos(\Omega_i t) = P_i \cos(\omega_i\tau_0 + \sigma_i\tau_2)\ , \quad i = 1,2,3 \tag{9}$$

i.e., the excitation frequencies are chosen to be close to the corresponding natural frequencies ω_i by means of detuning parameters σ_i, which are assumed to be of order ϵ^2.
Substituting Eq. (8) and corresponding derivatives and Eq. (9) in Eqs. (1-3), taking into account the chosen orders of magnitude of all the coefficients, and collecting terms of like powers of ϵ, we obtain, after some algebra, a perturbation hierarchy as a set of linear differential equations

$$m_0\, W_{1j,\tau_0\tau_0} - K_0\, W_{1j,xx} = \mathcal{P}_{1j}\ , \tag{10}$$

$$m_0\, W_{ij,\tau_0\tau_0} + K_S^{-1}\left(m_0\, m_2\, W_{ij,\tau_0\tau_0} - (m_0\, K_2 + m_2\, K_S)\, W_{ij,xx}\right)_{,\tau_0\tau_0} + K_2\, W_{ij,xxxx} = \mathcal{P}_{ij}\ , \qquad i = 2,3 \tag{11}$$

with the index j, that is the power of ϵ, spanning from 1 to 3.
Notice that terms $\mathcal{P}_{i1}$ vanish, $\mathcal{P}_{i2}$ depend on W_{i1}, and $\mathcal{P}_{i3}$ depend on W_{i1} and W_{i2}. Moreover, because of our assumptions on the orders of magnitude of $q_i(t)$, $\mathcal{P}_{23}$ is the first term in which an external load, namely $q_2(t)$, appears.
Although we consider only three time scales and neglect terms beyond the third order of ϵ in the expansions of unknowns, in multiple-scale approaches any number of scales and any order of ϵ can be considered. Indeed, the corresponding perturbation hierarchy is, at least in principle, simple to manage: starting by solving the first order problem, the right-hand side of second order problem can be computed; then, once second order problem is solved, the third order right-hand side is got, and so on. At any step proper solvability conditions must be met in order to avoid that resonant secular terms appear in the solution. However, typically, algebraic complexity allows to calculate a few terms of the expansion and convergence properties of the expansion remain unknown [8]. We should also point out that the asymptotic expansion introduced in Eqs. (8) for the unknown functions $w_i(x,t)$ gives an accurate representation of them for ϵ approaching zero.
After this brief introduction to the approach, the next step of this study will be to detail about frequency-response curves and time histories, with the aim to compare the behavior of the model we are dealing with to those of other nonlinear beam models available in the scientific literature [9, 10, 11].

Conclusions

The present contribution, which is part of an ongoing research focused on the analysis of a geometrically exact beam model with nonlinear constitutive relationships, reports on preliminaries of a multiple time scale expansion of the equations of motion. The next step, which is still in progress, will focus on approximate time histories, frequency-response curves, and comparison with other nonlinear beam models available in the scientific literature and with results obtained through numerical approaches as finite element or finite difference methods.

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