Exploration of edge states by bubbles in a constricted Hele-Shaw channel

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Summary. We study experimentally and numerically the propagation of an air bubble through a fluid-filled, geometrically-perturbed Hele-Shaw channel; a system which supports several stable modes of steady bubble propagation. During its transient evolution, a bubble may undergo several topological changes in the form of breakup and coalescence, depending on both initial conditions and control parameters. Long-term, either a single asymmetric or symmetric bubble is recovered or else multiple bubbles remain, whose relative separation increases with time. We explore how the transient and long-term evolution is controlled by edge states available to the bubble, which may change with each new topological configuration.

Introduction

Two-phase displacement flow in a confined geometry is a fundamental problem in fluid mechanics with applications in biomechanics, geophysics and industry. A canonical example is the viscous fingering instability which occurs when air displaces a viscous fluid in a Hele-Shaw channel of width much greater than its depth. An initially flat interface evolves into a single finger of air, symmetric about the centreline of the channel [1]. When the nondimensional driving flux is above a threshold, which depends on the roughness of the channel, this finger is observed to become unstable to tip-splitting leading to the emergence of complex patterns [2]. As the steady solution is linearly stable for all computed flow rates [3], these further instabilities must arise subcritically. For both air fingers (i.e. of semi-infinite extent) and bubbles (with fixed volume), models of the system also contain alternative, weakly-unstable families of multiple-tipped symmetric solutions and asymmetric solutions in addition to the linearly stable solution branch [4].
The system can be perturbed by introducing a small depth-perturbation to the channel cross section (see figure 1a, b) (termed a 'rail') which discourages propagation in the centre of the channel. The perturbed system is axially uniform and can support invariant propagation modes, which are broadly similar to those in the Hele-Shaw channel of rectangular cross-section. However, their stability is altered so that multiple linearly stable steady solutions occur for the same driving parameters (figure 1c). Hence, several long-term outcomes are possible for a bubble propagated from a centred initial position. For small flow-rates the bubble will readily settle on a stable off-rail state (see figure 1c) but for larger flow-rates the bubble shape will become increasingly deformed and a large range of transient behaviour is observed, including tip-splitting, and oscillatory behaviour [5]. Our previous studies have focused on a single-bubble configuration, where we showed that the bubble evolution is guided by transient exploration of the stable manifolds of weakly unstable edge states of the system [5]. In this paper, we explore in particular the behaviour at flow rates where the propensity for bubble breakup and topological changes can have a profound effect on the governing dynamics and lead to the bubble exploring multiple topological configurations before reaching its final state.

Experiment

The experimental channel is shown schematically in figure 1a,b. An air bubble of controlled volume is injected at one end of the channel and propagated by infusing silicone oil at a constant flow rate. We impose different initial conditions corresponding to bubbles of different initial widths by allowing a controlled relaxation period prior to flow initiation. The channel aspect ratio is fixed at 40, and the rail occupies 25% of the channel width and 2.4% of its height.

Numerical model

We use a depth-averaged model for bubble propagation in this large-aspect ratio channel. This model, which qualitatively captures all observed modes of finger propagation in the constricted Hele-Shaw channel [6], was recently extended to

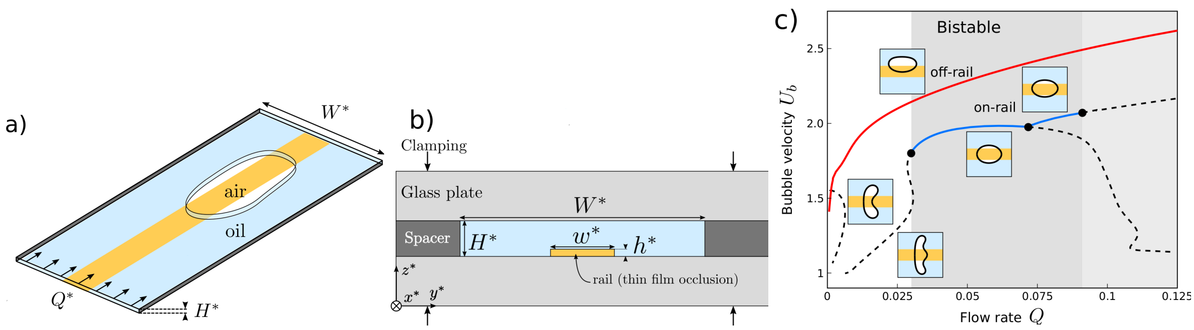

Figure 1: (a) Top-view of the experimental channel with an axially uniform, centred rail. (b) Cross-section of the channel. (c) Numerical bifurcation diagram depicting the steady modes of bubble propagation in terms of the bubble velocity as a function of the flow rate [5]. Stable and unstable modes are indicated by solid and dashed lines, respectively.

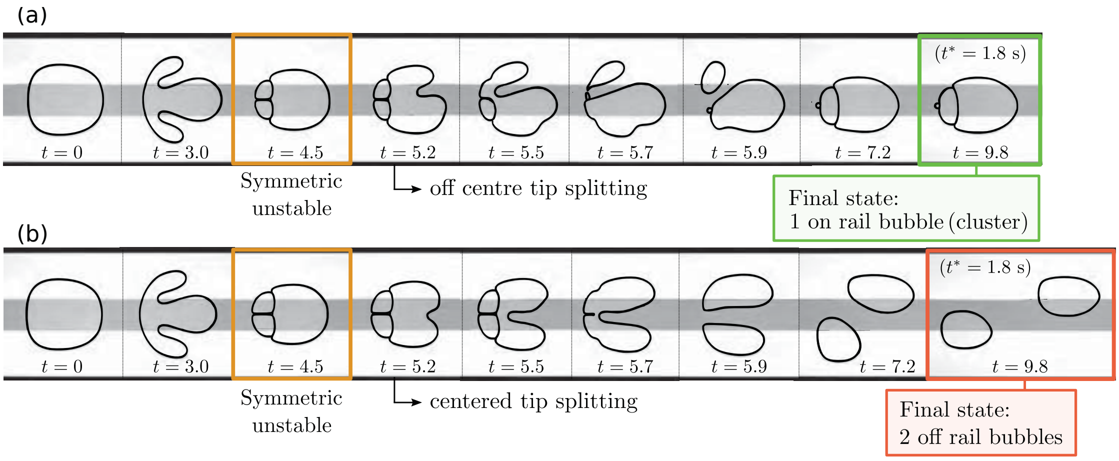

Figure 2: Examples of experimentally-observed transient bubble dynamics showing bubble breakup and recombination, leading to distinct outcomes (a) the compound bubble will eventually recombine to a single bubble and (b) two asymmetric bubbles which will eventually drift apart.

include bubble breakup and coalescence and to describe the motion of one or more bubbles. The use of a simplified model allows the computational effort to be concentrated on capturing the 2D interface as viewed in the experiment, enables a much wider range of computations, and also allows us to directly isolate different physical mechanisms. We solve our model equations using a finite-element method in the open-source library `oomph-lib`, and have access to steady solutions, bifurcation tracking, initial value problems, linear stability and weakly nonlinear stability analysis.

Results

An initially centred bubble may eventually reach the on-rail or off-rail single-bubble steady solutions, or else form two or more bubble fragments which drift away from each other as they propagate. These long-term outcomes can be reached via a surprisingly rich range of transient evolution scenarios, with the transient evolution depending on both control parameters and the initial bubble width. Only for the smallest flow rates does the bubble always remain simply connected throughout its evolution. Above a threshold flow rate, initially slender bubbles evolve towards on-rail steady states, while initially wide bubbles will break up into two or more separating bubbles. For example, figure 2 shows scenarios for two very similar bubbles. In each case the bubble first splits into three smaller bubbles, which then undergoes a complicated sequence of recombination and splitting, and eventually reach two different outcomes.

In order to explore the interactions between multiple bubbles and how these select the eventual fate, we concentrate on situations where the first topological change usually involves the bubble breaking into two near-equal size fragments. The subsequent evolution depends on the volume ratio selected by breakup and the relative position of each bubble, along with the driving parameters. We calculate steady solutions for the two-bubble configuration numerically using our depth-averaged model. Nearly all of these states are weakly unstable, but we find strong evidence that they act as edge states controlling whether the newly-formed bubbles remain on different sides of the rail (and hence usually separate) or whether they migrate across the rail to recombine.

Conclusion

This system exhibits rich transient evolutions, some of which involving bubble breakup into two or more bubbles, and can be followed by other topological reorganisations of the system. The problem is amenable to experiments and numerical simulations using a simplified model. We provide evidence that the dynamics of both single and multiple bubble systems are organised by weakly unstable solutions or “edge states”, and can calculate these using our simplified model. As the flow rate increases, transient bubble shapes are increasingly deformed and the number of breakups and recombinations undergone by the bubble increases. We suspect that this increase in complexity is due to a subcritical transition to disorder above a threshold that depends on the roughness of the occlusion, reminiscent of the transition to turbulence in shear flow.

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