Freezing of Unsteady Nonlinear Waves over an uneven bottom by Phase-Space Manipulation

A. N. Gomel*, D. Eeltink*, M. Brunetti*, J. Kasparian*, A. Chabchoub†, S.Trillo‡ and A.Armaroli§
*Institute of Environmental Science, University of Geneva, Bd. Carl Vogt 66, 1205 Geneva, Switzerland
†School of Civil Engineering, The University of Sydney, Australia
†Disaster Research Institute Hakubi Center for Advanced Research, Kyoto University, Japan
‡Deparment of Engineering, University of Ferrara, Via Saragat 1, Italy
§PhLAM, Université de Lille, 50 avenue Halley, Villeneuve d'Ascq, France

Summary. We develop two frameworks for "freezing" modulationally unstable wave packets of gravity waves over an uneven bottom modeled by a finite-depth third-order non-linear Schrödinger equation. We compare with experimental results in a 30 m wavetank featuring a sharp depth transition and propose a theoretical route for stabilizing of the modulated wave packets over an "adaibatic" depth variation.

Theoretical framework and experimental setup

We consider the stabilization of modulationally unstable wave packets within the framework of the nonlinear Schrödinger equation (NLS) in variable water depth [1].

$$i\frac{\partial U}{\partial \xi} + \alpha(kh)\frac{\partial^2 U}{\partial \tau^2} - \beta(kh)|U|^2\,U = -i\mu_0\,\frac{\partial(kh)}{\partial \xi}\,U \qquad (1)$$

This is possible thanks to the simultaneous dependence of the parameters α and β in Eq. (1) on the adimentional depth (kh), and has also been verified in an optical fiber experiment [4], where there is even more freedom to change the parameters.

We provide two frameworks for the understanding and development of the stabilization, *or "freezing"*, of highly modulated waves. The first theoretical framework is to connect a solution related to modulationally unstable waves, such as the Akhmediev breather (AB), to a solution like the dnoidal function which is related to a stable wavepacket. The second consists in reducing the phase space to that of a carrier wave of frequency ω_0 and the first two sidebands at $\omega_0 \pm \Omega$ (the so-called three wave picture). In this context, "freezing" the wave packet corresponds to an expansion of a homoclinic orbit and its transformation into an elliptic fixed point through the simultaneous variation of the NLS parameters α and β. We experimentally demonstrate this process in a $30 \times 1\ m^2$ water wave tank with a fake bottom floor as shown in figure 1 (a), and provide a rigorous theoretical description of this process for a sharp change in bathymetry [3]. We also provide a stabilization route when the bathymetry change is very slow [2]. The theoretical predictions in the three-wave picture and the measurements show that the relative phase among the side-bands locks to π and their relative amplitude oscillates around a finite value (fig 1, panels on the right). As shown in figure 1, apart from a 10% conversion to higher-order side-bands, this implies that the breathing stage of modulation instability (MI) is indeed frozen. We confirm that this complex wave dynamics is robust and such control of MI processes is feasible in a realistic experimental system. Our results highlight the influence of topography and how waveguide properties can influence and manipulate the lifetime of nonlinear and extreme waves.

Conclusions

We study the nonlinear stage of evolution of modulational instability in surface gravity waves over a water body of increasing depth. We show that this stage can be stabilized and results in a uniform train of pulses on a background.

The initial condition does not need to be restricted to an exact NLS solution (as ABs), since we have shown that freezing occurs also in a three-wave system.

We have found a theoretical condition to dynamically stabilize unstable nonlinear waves. While the approach applies to any system described by the NLS equation, and could therefore be easily generalized to other dynamical models, we have experimentally confirmed our finding for the specific case of wave hydrodynamics. A sharp change in water depth simultaneously modifies the dispersion and nonlinearity experienced by surface gravity wave packets, thus dramatically modifying their dynamical behaviour. In the case of ABs, the separatrix expands and ends up enclosing the system trajectory, which is stabilized around an elliptic fixed point. This jump can be described as the optimal matching of an initial AB solution to a steady dnoidal solution of the universal NLS equation, illustrating the generality of this wave control process.

Although the flexibility available to vary parameters in the hydrodynamics of surface water-waves is much less than in other physical systems, such as optical fibers, our results help to clarify the possibility to dynamically control the breathing evolution of water wave-packets and to understand the impact of bathymetry on the persistence (or lifetime) of rogue waves.

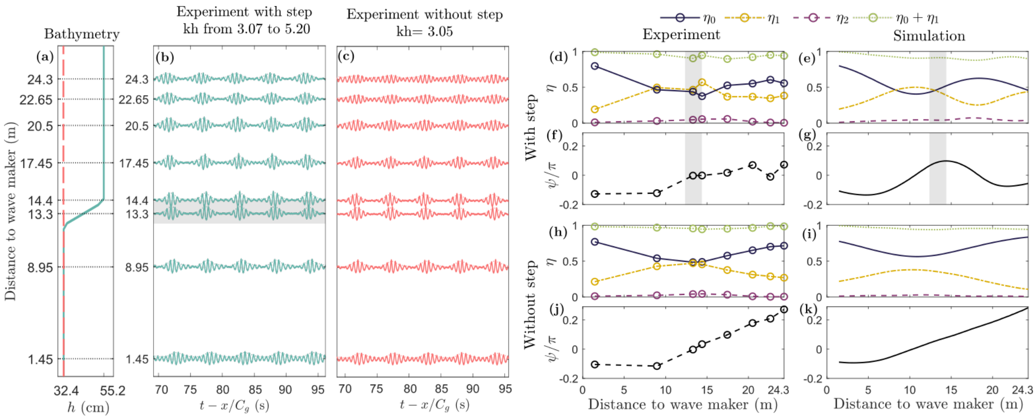

Figure 1: Left: (a) Water wave flume with artificial floor setup (cyan line), and the constant floor setup (dashed coral line). (b) Wave height at each recorded position for the experiment with variable bathymetry, multiplied by a factor 20; the gray stripe indicates the position of the step. (c) Wave height at each recorded position for the experiment with constant bathymetry, multiplied by a factor 20. (d)–(k) Sideband evolution of the AB-type surface water wave over the adopted bathymetry with the depth step (d),(e),(f),(g), and the constant flat bottom h_0 (h),(i),(j),(k). Here the initial condition for the envelope is of the form $U(\xi, \tau = 0) = u_0 e^{i\varphi_0} + u_1 e^{i\Omega\varphi_1} + u_{-1} e^{-i\Omega\varphi_{-1}}$, the sideband fraction $\eta_j = |u_j|^2/(|u_0|^2 + |u_1|^2 + |u_{-1}|^2)$ and the relative phase $\psi = (\varphi_1 + \varphi_{-1})/2 - \varphi_0$. (d)–(h) Sideband dynamics as identified from the eight gauge measurements, connected by a linear interpolation; (g)–(i) corresponding NLS equation-simulated evolution. (d),(e),(h),(i) Sideband fractions η_0, η_1, η_2 of modes at detuning 0 (carrier), Ω, and 2Ω, respectively. (f),(g),(j),(k) Phase ψ of first-order sidebands (modes at $\pm\Omega$) relative to the carrier frequency .

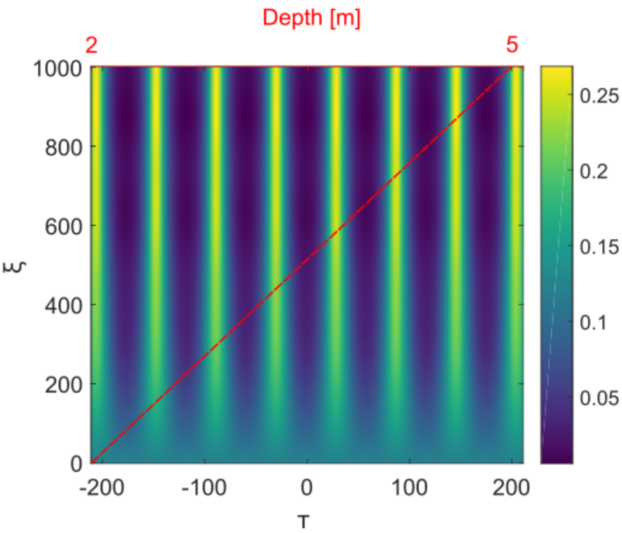

Figure 2: Evolution of the envelope amplitude (shown in meters on the colorbar) from a perturbated plane wave initial condition to a stabilized modulated wave packet over a constant slope from 2 m to 5 m.

References

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