Period approximation for nonlinear oscillators with Carleman linearization

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Summary. A method for period approximation of nonlinear oscillators using Carleman linearization is presented. The approximation of the amplitude-dependent period of a given nonlinear oscillator is calculated and compared to exact period values.

Introduction

Calculation of the period of a nonlinear oscillator is an important engineering problem. In this paper, a versatile tool, the so-called Carleman linearization [1] is used to obtain the solutions of nonlinear oscillator $\ddot{x} + f(x,\, \dot{x}) = 0$ and its exact amplitude-dependent period is approximated. An example of such nonlinear oscillator is

$$\ddot{x} + (1 + \dot{x}^2)x = 0, \quad x(0) = A, \quad \dot{x}(0) = 0. \tag{1}$$

Mickens et al. [2] derived the formula of the exact period of the oscillator (1):

$$T_{exact}(A) = 4A \int_0^1 \frac{\mathrm{d}z}{\sqrt{\exp(A^2(1-z^2)) - 1}}. \tag{2}$$

An approximation of the exact period in case of small values of A is given in [3] as

$$T_{approx}(A) \approx 2\pi \left(1 - \frac{1}{8}A^2 + \frac{1}{256}A^4 + \frac{5}{6144}A^6 - \frac{7}{262144}A^8 + \dots\right). \tag{3}$$

Small and large amplitude periodic orbits of Eq. (1) were investigated by Kalmár-Nagy and Erneux in [4].

Carleman linearization and period approximation

Eq. (1) can be written as

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -(1 + x_2^2)x_1. \end{aligned} \tag{4}$$

By introducing the notation

$$\mathbf{x}^{[j]} = (x_1^j,\ x_1^{j-1}x_2,\ x_1^{j-2}x_2^2,\ \dots,\ x_1^2x_2^{j-2},\ x_1x_2^{j-1}, x_2^j)^{\mathrm{T}}, \quad j = 1,\, \dots,\, n, \tag{5}$$

and applying Carleman linearization [1] Eq. (4) is recast as

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \mathbf{x}^{[1]} \\ \mathbf{x}^{[2]} \\ \mathbf{x}^{[3]} \\ \vdots \\ \mathbf{x}^{[n-2]} \\ \mathbf{x}^{[n-1]} \\ \mathbf{x}^{[n]} \end{pmatrix}}_{\dot{\mathbf{y}}_n} = \underbrace{\begin{pmatrix} \mathbf{B}_{1,1} & \mathbf{0} & \mathbf{B}_{1,3} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{2,2} & \mathbf{0} & \ddots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \dots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{B}_{n-2,n-2} & \mathbf{0} & \mathbf{B}_{n-2,n} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{B}_{n-1,n-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{B}_{n,n} \end{pmatrix}}_{\mathbf{C}_n} \underbrace{\begin{pmatrix} \mathbf{x}^{[1]} \\ \mathbf{x}^{[2]} \\ \mathbf{x}^{[3]} \\ \vdots \\ \mathbf{x}^{[n-2]} \\ \mathbf{x}^{[n-1]} \\ \mathbf{x}^{[n]} \end{pmatrix}}_{\mathbf{y}_n}, \tag{6}$$

$$\mathbf{y}_0(A) = \left(A,\, 0,\, A^2,\, 0,\, 0,\, A^3,\, 0,\, 0,\, 0, \dots,\, A^n,\, \dots,\, 0\right)^{\mathrm{T}},$$

where $\mathbf{C}_n$ denotes the Carleman matrix of order n and $\mathbf{y}_0$ is the vector of initial conditions. The matrices $\mathbf{B}_{j,j}$, $j = 1,\, \dots,\, n$ and $\mathbf{B}_{k,k+2}$, $k = 1,\, \dots,\, n-2$ are constructed as follows

$$\mathbf{B}_{j,j} = \begin{pmatrix} 0 & -j & & & & & \\ 1 & 0 & -j+1 & & & \mathbf{0} & \\ & 2 & 0 & \dots & & & \\ & & \dots & 0 & -2 & & \\ & \mathbf{0} & & j-1 & 0 & -1 \\ & & & & j & 0 \end{pmatrix}, \quad \mathbf{B}_{k,k+2} = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & -2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -k & 0 \end{pmatrix}. \tag{7}$$

The approximation of the solution of Eq. (4) is written as

$$\widehat{x}_1(t) = \mathbf{e}_1^{\mathrm{T}} e^{\mathbf{C}_n t}\mathbf{y}_0(A), \quad \widehat{x}_2(t) = \mathbf{e}_2^{\mathrm{T}} e^{\mathbf{C}_n t}\mathbf{y}_0(A), \tag{8}$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are standard basis vectors. The approximate solution $\widehat{x}_2(t)$ can be written as power series of A, i.e.,

$$\widehat{x}_2(t) = \sum_{k=1}^{n} \tilde{x}_{2,k}(t)\frac{A^k}{k!}. \tag{9}$$

The period $T(A)$ of system (4) is approximated based on [5] as

$$T(A) = 2\pi + \Delta T(A) = 2\pi + \sum_{k=1}^{n} A^k \widetilde{T}_k. \tag{10}$$

The coefficients $\tilde{x}_{2,k}(t)$ in Eq. (9) at the period $T(A)$ are expressed as

$$\tilde{x}_{2,k}(T(A)) = \tilde{x}_{2,k}(2\pi + \Delta T(A)) = \tilde{x}_{2,k}(2\pi) + \sum_{m=1}^{n} \tilde{x}_{2,k}^{(m)}(2\pi)\frac{\Delta T(A)^m}{m!}, \tag{11}$$

where $\tilde{x}_{2,k}^{(m)}$ denotes the mth derivative. Since Eq. (4) is a conservative system [4], $\widehat{x}_2(T(A)) = 0$ must hold . Balancing the terms

$$\begin{aligned} A &: 0 = \tilde{x}_{2,1}(2\pi), \\ A^2 &: 0 = \tilde{x}_{2,2}(2\pi) + \tilde{x}'_{2,1}(2\pi)\widetilde{T}_1, \\ A^3 &: 0 = \tilde{x}_{2,3}(2\pi) + \tilde{x}'_{2,2}(2\pi)\widetilde{T}_1 + \tilde{x}'_{2,1}(2\pi)\widetilde{T}_2 + \tilde{x}''_{2,1}(2\pi)\frac{\widetilde{T}_1^2}{2!}, \\ &\vdots \end{aligned} \tag{12}$$

where the primes mean derivation. System (12) is solved for the unknown $\widetilde{T}_k$'s.
Using the Carleman linearization of order $n = 9$ of system (4) the approximation of the period reads as

$$T(A) \approx 2\pi\left(1 - \frac{1}{8}A^2 + \frac{1}{256}A^4 + \frac{5}{6144}A^6 - \frac{7}{262144}A^8\right), \tag{13}$$

the same as in Eq. (3). The following table shows some numerical results in case of $n = 5,\ 7,\ 9$ order Carleman matrices.

		$n=5$		$n=7$		$n=9$	
A	$T_{exact}(A)$	$T(A)$	Rel. error [%]	$T(A)$	Rel. error [%]	$T(A)$	Rel. error [%]
0.01	6.2831	6.2831		6.2831		6.2831	
0.1	6.2753	6.2753		6.2753		6.2753	
1	5.5272	5.5223	0.088	5.5274	0.004	5.5273	0.001
1.5	4.6903	4.6403	1.065	4.6985	0.176	4.6942	0.085
2	3.7613	3.5343	6.036	3.8615	2.644	3.8186	1.522
2.2	3.4131	3.0568	10.44	3.6366	6.547	3.5445	3.850

Conclusions

A new way of calculation for the period of a nonlinear oscillator was introduced. Approximate period values of a given oscillator were calculated and compared to exact ones. We conclude, in the case of small amplitudes, Carleman linearization can be used for approximation of the period of a nonlinear oscillator.

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