Time-domain Wave Propagation in Rigid Porous Media using Equivalent Fluid Model with a Quadratic Nonlinearity

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Summary. The acoustic properties of rigid porous media can be described by the equivalent fluid model (EFM) in the frequency domain, involving complex-valued functions. These physical quantities can be irrational, which leads to fractional derivatives in the time domain. Besides, this model is built with a constant flow resistivity, which is known to grow linearly with the flow velocity in the Forchheimer regime. Hence, a correction on the EFM is made according to the Darcy-Forchheimer law, leading to a more general model with an additional nonlinear term. Here, an approach is presented to formulate the EFM equations with the Forchheimer's correction in the time domain, where the fractional derivatives described by causal convolution are approximated by additional differential equations. It results in a nonlinear system on which an energy-based analysis is performed to ensure its stability under suitable conditions.

Nonlinear equivalent fluid model equations

When the material can be assumed rigid, acoustic wave propagation in porous media can be well represented by:

$$\begin{cases} \rho_0\alpha_\infty\,\partial_t\mathbf{u} + \boldsymbol{\nabla}p + M\,\mathbf{u} + N\,(g\star\partial_t\mathbf{u}) \;=\; -\,F_\xi\,|\mathbf{u}|\mathbf{u}\,, & \text{(1a)}\\ \chi_0\,\partial_t p + \boldsymbol{\nabla}\cdot\mathbf{u} + (\gamma-1)\,N'\,(h\star\partial_t p) \;=\; 0\,, & \text{(1b)}\end{cases}$$

where ρ_0 is the ambient fluid density, α_∞ the high frequency limit of the material dynamic tortuosity, χ_0 the ambient fluid adiabatic compressibility, and γ the heat capacity ratio; $\mathbf{u}$ and p are the particle velocity and pressure, respectively. Factor M is defined through Darcy's law. The two terms involving the kernel functions g and h and the convolution operator $\star$ account for viscous and thermal losses in the porous medium. Definitions of M, N, N', g and h are based on the dynamic tortuosity α and the dynamic compressibility β, for which complex-valued analytical models exist, such as the JCAPL model [1], presented below with the same parameters:

$$\hat{\alpha}(s) = \alpha_\infty\left[1+\frac{M}{s}+N\frac{\sqrt{1+\dfrac{s}{L}}-1}{s}\right], \quad (2) \qquad \hat{\beta}(s) = \gamma-(\gamma-1)\left[1+\frac{M'}{s}+N'\frac{\sqrt{1+\dfrac{s}{L'}}-1}{s}\right]^{-1}, \quad (3)$$

where $\hat{f}$ denotes the Laplace transform of f, s is the Laplace variable and all the coefficients are defined from physical quantities, detailed in [1]. Lastly, an additional nonlinear term to the initial EFM is taken into account in the right-hand side of (1a), representing the inertial effects induced by high amplitude waves travelling in the material. It comes from the nonlinear Fochheimer equation [2] where the multiplying factor F_ξ depends on the Forchheimer nonlinearity parameter ξ, computed in experimental studies [3, 4] by considering a linear relation between the total flow resistivity σ and the velocity amplitude:

$$\sigma = \sigma_0\,(1+\xi\phi|\boldsymbol{u}|)\;, \tag{4}$$

where ϕ is the porosity and σ_0 is the static flow resistivity. This relation is known as Forchheimer's correction.

Nonlinear equivalent fluid model with additional differential equations

In order to build an efficient numerical scheme for (1), a reformulation of the equations is performed using the additional differential equations (ADE) method [5]. The irrational functions $\hat{g}$ and $\hat{h}$ are first approximated by rational functions $\hat{G}$ and $\hat{H}$, written as multipole models and parameterized by a set of real or complex conjugate weights and poles:

$$\hat{G}(s) = \sum_{k=1}^{K}\frac{r_k}{s-s_k}\;, \quad (5) \qquad \hat{H}(s) = \sum_{k=1}^{K'}\frac{r'_k}{s-s'_k}\;. \quad (6)$$

All the parameters r_k, r'_k, s_k and s'_k are here assumed to be real and can be computed using different available methods [6, 7, 8]. Then, using (5) and (6) in system (1) written in the Laplace domain, leads to the following system:

$$\begin{cases} \rho_0\alpha_\infty\,s\,\hat{\mathbf{u}} + \boldsymbol{\nabla}\hat{p} + M\,\hat{\mathbf{u}} + N\displaystyle\sum_{k=1}^{K}\left(r_k+\frac{r_k s_k}{s-s_k}\right)\hat{\mathbf{u}} = -\,F_\xi\widehat{|\mathbf{u}|\mathbf{u}}\,,\\ \chi_0\,s\,\hat{p} + \boldsymbol{\nabla}\cdot\hat{\mathbf{u}} + (\gamma-1)N'\displaystyle\sum_{k=1}^{K'}\left(r'_k+\frac{r'_k s'_k}{s-s'_k}\right)\hat{p} = 0\,.\end{cases} \tag{7}$$

In order to express the discrete equivalent in the time domain, the inverse Laplace transform is applied, leading to a new set of equations with causal convolutions which are computed by introducing additional variables as follows:

$$\boldsymbol{\varphi_k}(t) = (e_{s_k}\star\mathbf{u})(t)\;, \quad (8) \qquad \psi_k(t) = (e_{s'_k}\star p)(t)\;, \quad (9)$$

where $e_x : t \to e^{xt}\mathrm{H}(t)$ and H is the Heaviside function.

Each of these additional variables is solution to a first-order ordinary differential equation and can be computed with a standard time-integration scheme. Finally, the global system of equations in the time domain reads:

$$\begin{cases} \rho_0\alpha_\infty\,\partial_t\mathbf{u} + \boldsymbol{\nabla} p + \left(M + N\displaystyle\sum_{k=1}^{K} r_k\right)\mathbf{u} + N\displaystyle\sum_{k=1}^{K} r_k s_k \boldsymbol{\varphi_k} \;=\; -F_\xi|\mathbf{u}|\mathbf{u}\ , \\ \chi_0\,\partial_t p + \boldsymbol{\nabla}\cdot\mathbf{u} + (\gamma-1)\left(N'\displaystyle\sum_{k=1}^{K'} r'_k\right)p + (\gamma-1)\,N'\displaystyle\sum_{k=1}^{K'} r'_k s'_k \psi_k \;=\; 0\ \ , \\ \partial_t\boldsymbol{\varphi_k} = s_k\boldsymbol{\varphi_k} + \mathbf{u} \qquad\qquad (\forall k\in[1,K])\ , \\ \partial_t\psi_k = s'_k\psi_k + p \qquad\qquad (\forall k\in[1,K'])\ . \end{cases} \tag{10}$$

Note that in (10), there are no spatial derivatives in the ADE. Hence, when the system is discretized with a numerical scheme based on fluxes, these fluxes depend on the velocity and pressure variables, but not on the additional variables. Consequently, the problem to solve at each mesh interface does not grow with the number of additional variables.

Stability analysis

Hereafter, a stability analysis of system (10) is performed thanks to the energy functional defined below:

$$\mathcal{E}(t) := \frac{1}{2}\left(\rho_0\alpha_\infty\int_\Omega \|\mathbf{u}\|^2\ \mathrm{d}x - N\sum_{k=1}^{K} r_k s_k\int_\Omega \|\boldsymbol{\varphi_k}\|^2\ \mathrm{d}x\right) + \frac{1}{2}\left(\chi_0\int_\Omega p^2\ \mathrm{d}x - (\gamma-1)\,N'\sum_{k=1}^{K'} r'_k s'_k\int_\Omega \psi_k^2\,\mathrm{d}x\right), \tag{11}$$

the derivative of which is

$$\begin{aligned} \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}(t) = \quad & -M\int_\Omega \|\mathbf{u}\|^2\ \mathrm{d}x - F_\xi\int_\Omega |\mathbf{u}|\,\|\mathbf{u}\|^2\ \mathrm{d}x - N\sum_{k=1}^{K} r_k\int_\Omega \|\partial_t\boldsymbol{\varphi_k}\|^2\ \mathrm{d}x \\ & -(\gamma-1)\,N'\sum_{k=1}^{K'} r'_k\int_\Omega (\partial_t\psi_k)^2\ \mathrm{d}x - \int_{\partial\Omega}(p\,\mathbf{u})\cdot\mathbf{n}\,\mathrm{d}\sigma\ . \end{aligned} \tag{12}$$

From (11) and (12), combined with the fact that ρ_0, α_∞, χ_0 and $(\gamma-1)$ are necessarily positive for porous media, as well as the parameters M, N, N', F_ξ, we impose the following sufficient conditions to ensure stability:

$(\mathbf{C_1})$ the weights $(r_k, r'_k)_k$ are positive, $(\mathbf{C_2})$ the poles $(s_k, s'_k)_k$ are negative.

When both these conditions are met, $\mathcal{E}$ is a positive-definite functional, which is decreasing in time without external contributions at the boundary $\partial\Omega$ of the domain Ω.

Conclusion

This work formulates the nonlinear EFM equations for porous media in the time domain, with the diffusive part of the complex-valued functions α and β described by a set of weights and poles. In addition, a proof of stability is given under conditions on the sign of these, conditions satisfied by the JCAPL model for a realistic range of physical parameters. A numerical study will be carried out in order to check the theoretical results and to investigate the effects of nonlinearity.

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