# Numerical continuation of periodic solutions with constraints: application to a physical model of wind musical instrument

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*Summary*. Numerical continuation using the Asymptotic Numerical Method (ANM), together with he Harmonic Balance Method (HBM), allows to follow the periodic solutions of non-linear dynamical systems such as physical models of wind instruments. This has been successfully applied to practical problems such as the categorization of musical instruments from the calculated bifurcation diagrams [2]. Nevertheless, one problem often encountered concerns the uncertainty on some parameters of the model, the values of which are set arbitrarily because too difficult to measure experimentally. In this work we propose a novel approach where constraints based on experimental measurements are added to the system, as well as the uncertain parameters of the model relaxed. This approach allows the continuation of the periodic solution with constraints to be performed, together with the calculation of the variation of the relaxed parameters along the solution branch. A successful application of this technique to a physical model of a trumpet is presented in this paper.

## Physical model of the {player-trumpet} system and continuation

We consider a one-dimensional lip model, coupled to the resonator impedance described by a series of complex modes similar to what is proposed in [2]. The coupling between the mechanical oscillator and the acoustic resonator is achieved by a stationary Bernoulli flow equation, considering turbulent mixing in the mouthpiece with no pressure recovery. The mechanical and acoustic equations are given in system 1, where $y$ is the vertical lip position ($y_0$ is the lip position at rest), $\omega_l$, $Q_l$, $\mu_l$ and $b$ the lip mechanical parameters, $s_k$ and $C_k$ with $k \in [1, N]$ the modal parameters of the $N$ resonances of the acoustic impedance of the instrument, $Z_c$ the characteristic impedance, $u$ the volume flow, $p$ the downstream pressure at the input of the instrument (in the mouthpiece), and $p_0$ the upstream (mouth) static pressure.

$$\begin{cases} \ddot{y}(t) + \frac{\omega_l}{Q_l}\dot{y}(t) + {\omega_l}^2(y(t) - y_0) = \frac{1}{\mu_l}(p_0 - p(t)) \\ \dot{p}_k(t) = Z_c C_k u(t) + s_k p_k(t), \forall k \in [1, N] \end{cases} \tag{1}$$

with $p(t) = 2\sum_{k=1}^{N} \Re(p_k(t))$ and $u = \sqrt{\frac{2|p_0-p|}{\rho}} b \cdot sign(p_0 - p) \cdot \theta(y)$, where $\theta(y) = \frac{|y|+y}{2}$, $b$ is the lip width and $\rho$ is the air density.

The case of a negative opening of the lips is managed by introducing the function $\theta(y)$ which enforces $u = 0$ if $y < 0$. The modal parameters of the $N$ modes of the impedance are extracted from the measured input impedance using the high resolution method ESPRIT [3]. In this model, the values of the lip parameters are critical but particularly difficult to set, as it is extremely difficult to evaluate them experimentally.
We choose to work with the Asymptotic Numerical Method (ANM) implemented in the software MANLAB [4]. This method is based on the expansion of the solutions under the form of truncated Taylor series, providing analytical formulations of the branch of solution. Recently, this method has been associated to the Harmonic Balance Method (HBM) for the search of periodic solutions of oscillating systems [5].
One requirement of MANLAB relies on the recast of nonlinearities of the model into, at most, quadratic nonlinearities. The complete quadratic dimensionless model can be found in [2].

## Continuation with constraints

Two constraints are introduced as follows:

$$\|\tilde{p}\|_{L2} = S\gamma + I, \tag{2}$$

where $\gamma = p_0/P_M$ is the dimensionless mouth pressure with $P_M = \mu_l {\omega_l}^2 y_0$, $S$ and $I$ are constant values, and $\|\tilde{p}\|_{L2} = 2\left\|\sum_{k=1}^{N} \Re(\tilde{p}_k)\right\|_{L2}$.
The second constraint simply writes as follows:

$$f_0 = F, \tag{3}$$

where $F$ is a constant value.
Adding two equations to the system requires two parameters of the model to be relaxed, that is two unknowns to be introduced. We choose to relax $Q_L$ and $\zeta = Z_c b y_0 \sqrt{\frac{2}{\rho P_M}}$ ($\zeta$ can be seen as an "embouchure" parameter). This requires to recast the system of equations in order to preserve the quadratic property of the model.

## Results and conclusions

Figure 1 shows the result of continuation with constraints applied to the physical model described in previous section. In the bottom plot, the evolution of $||\tilde{p}||_{L2}$ with respect to $\gamma$ measured on a trumpet player during a slow crescendo-decrescendo maneuver is represented in red. A linear fit of the red curve is applied, which gives $S$ and $I$ (Eq. 2) and defines the constraint on $||\tilde{p}||_{L2}$. The constraint on $f_0$ is such as it remains constant and equal to the value at the initial calculation point (about the playing frequency of a Bb4).

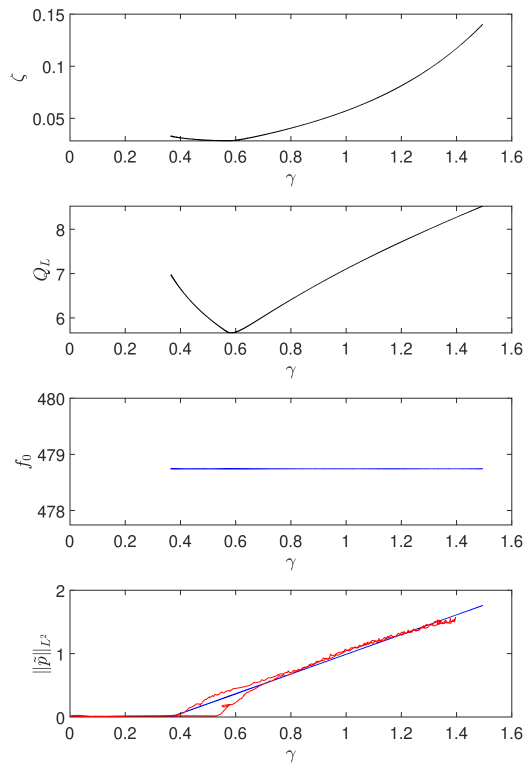


Figure 1: Results from continuation with constraints. Evolution of $\zeta$, $Q_L$, $f_0$ and $||\tilde{p}||_{L2}$ with respect to the dimensionless mouth pressure $\gamma$. In red is the evolution of $||\tilde{p}||_{L2}$ with respect of $\gamma$ measured on trumpet player during a slow crescendo-decrescendo maneuver.

It can be seen that the two constraints are well respected: $f_0$ is constant with respect to $\gamma$, and $||\tilde{p}||_{L2}$ evolves linearly with respect to $\gamma$ (the solution branches, in blue, are superimposed with the target constraint on Fig. 1 bottom plots). The stability of the branch was computed and the branch was found stable across the whole range of $\gamma$. The variations of $\zeta$ and $Q_L$ are represented on the two top plots. Significant variations of these two variables are observed, showing the importance of adapting these parameters in order to match the constraints.

These results highlight the ability of the ANM to calculate the evolution of some parameters of the model while applying some mathematical constraints to the output of the continuation calculation. By defining these constraints from experimental data, this approach can be seen as an inversion method, allowing to retrieve the parameter values of the model necessary to achieve a given performance (playing a crescendo-decrescendo at completely constant playing frequency).

This method then shows great perspectives for the parametrisation of physical models of brass instruments, as well as for objective comparison of brass instruments.

## References

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