On the solution of the Mathieu equation with multiple harmonic stiffness: parametric amplification for constant and harmonic forcing.

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Summary. In this abstract, we focus on the study of a single degree-of-freedom Mathieu-type differential equation, which can be found in engineering examples such as in the modelling of geared systems. We consider that the variable part of the stiffness is composed of several harmonic components and that the forcing terms contain a constant component along with an harmonic component. The phenomenon of parametric amplification is described in the case of a single harmonic force, as well as in the case of a purely constant force. The combination of both forces is also considered in the study

Introduction

This abstract presents a study of a Mathieu type equation arising, for example, when modelling the meshing of two gears mounted on rigid shafts [1] or when studying a pendulum attached to a moving point [2]. The equation represents the motion of a (damped and forced) single degree-of-freedom attached to the ground through a periodically time varying stiffness and is given by the following:

$$\ddot{x} + \mu\dot{x} + [\omega_0^2 + g(t)]x = h(t), \tag{1}$$

where $x(t)$ represents the variable under interest, μ is the damping coefficient, ω_0 is the natural angular frequency, $g(t)$ is the periodic time varying stiffness (with zero mean) and $h(t)$ is the external force.
In the following, we will assume that the stiffness $g(t)$ is periodic with fundamental pulsation Ω, such that it can be represented by its Fourier series: $g(t) = \sum_{p\neq 0} \widehat{\chi}_p e^{ip\Omega t}$, where $\widehat{\chi}_p \in \mathbb{C}$ is the complex amplitude of the p-th harmonic of the stiffness. The external force $h(t)$ will be considered to be the sum of a constant part h_0 and an harmonic part $h_n(t)$ at a given order $n \in \mathbb{N}$: $h(t) = h_0 + (\widehat{h}_n e^{in\Omega t} + c.c.)$, where $\widehat{h}_n \in \mathbb{C}$ is the complex amplitude of the n-th harmonic of the external force, and $c.c.$ stands for complex conjugate.
After introducing some scaling using a small parameter ϵ, the multiple scale method [3] can be used to derive approximated solutions to Eq.(1) under the form $x(t) = x_0(t) + \epsilon x_1(t) + ...$ In this study, one supposes that the damping coefficient (μ), the variable part of the stiffness (g) and the variable part of the force (h_n) are of order ϵ (all the other terms remaining at order 1). The application of the multiple scale method up to order one, leads to two linear differential equations for x_0 and x_1, and the cancellation of the secular terms inside the second equations allows to identify the coupling that can occurs between the different harmonics of the stiffness and the force, along with the computation of the approximated solution $x_0(t)$.
In this study, we consider the solution to Eq.(1) for three different kinds of forcing: i) harmonic forcing only ($h_0 = 0$)), ii) constant forcing only ($h_n = 0$) and iii) combination of constant and variable forcing. In each cases, we describe the phenomenon of parametric amplification exposed by the multiple scale method and we use numerical methods to validate our results. Note that in this study, one will consider that the damping coefficient μ is sufficiently high to avoid parametric instabilities.

Results

Harmonic forcing only

Here we first consider that the system is forced harmonically ($h_0 = 0$ and $\widehat{h}_n \neq 0$) close to its resonance ($n\Omega \approx \omega_0$). The cancellation of the secular terms shows that, at first order in ϵ, the $p = 2n$-th harmonic of the stiffness interacts with the (n-th) harmonic of the force, leading to the following form for the approximated solution:

$$x(t) \approx \left(\frac{\widehat{N}(\Omega)\widehat{h}_n + \widehat{h}_n^* \widehat{\chi}_{2n}}{D(\Omega)} e^{in\Omega t} + c.c. \right), \tag{2}$$

with $\widehat{N}(\Omega) \in \mathbb{C}$ and $D(\Omega) \in \mathbb{R}$. It can be seen that, for a given Ω (e.g. $n\Omega = \omega_0$), the amplitude of the solution can be made maximum (resp. minimum) if the phase between the complex numbers $\widehat{N}(\Omega)\widehat{h}_n$ and $\widehat{h}_n^* \widehat{\chi}_{2n}$ is zero (resp. is π). This phenomenon is known as the parametric amplification effect: by choosing conveniently the complex amplitude $\widehat{\chi}_{2n}$ on can create an amplification, or an attenuation of the vibration amplitude for a given excitation frequency (usually around resonance) [4, 5].

Constant forcing only

Here, we consider that the system is only statically forced ($h_0 \neq 0$ and $\widehat{h}_n = 0$), and that the fundamental pulsation of the periodic stiffness is such that $p\Omega \approx \omega_0$ (for a given $p \in \mathbb{N}$ such that $\widehat{\chi}_p \neq 0$). The cancellation of the secular terms shows that, at first order in ϵ, the p-th and the $2p$-th harmonic of the stiffness interacts together, leading to the following form for the approximated solution:

$$x(t) \approx \frac{h_0}{\omega_0^2}\left[1 + \left(\frac{\widehat{N}(\Omega)\widehat{\chi}_p + \widehat{\chi}_p^*\widehat{\chi}_{2p}}{D(\Omega)} e^{ip\Omega t} + c.c.\right)\right], \tag{3}$$

again with $\widehat{N}(\Omega) \in \mathbb{C}$ and $D(\Omega) \in \mathbb{R}$. It can be seen that, for a given Ω (e.g. $p\Omega = \omega_0$), the amplitude of variable part of the solution can be made maximum (resp. minimum) if the phase between the complex numbers $\widehat{N}(\Omega)\widehat{\chi}_p$ and $\widehat{\chi}_p^*\widehat{\chi}_{2p}$ is zero (resp. is π). This phenomenon appears to be very similar to the parametric amplification effect presented before: adjusting the phase between the p-th and $2p$-th harmonic of the stiffness can lead to amplification or attenuation of the vibration level (see Fig.1). To our knowledge, this phenomenon is not often refereed to (one explanation might be that, most of the time, researchers consider Mathieu's equation with only a single harmonic variable stiffness along with an harmonic forcing only), and we think it might be useful to understand and/or design gear systems.

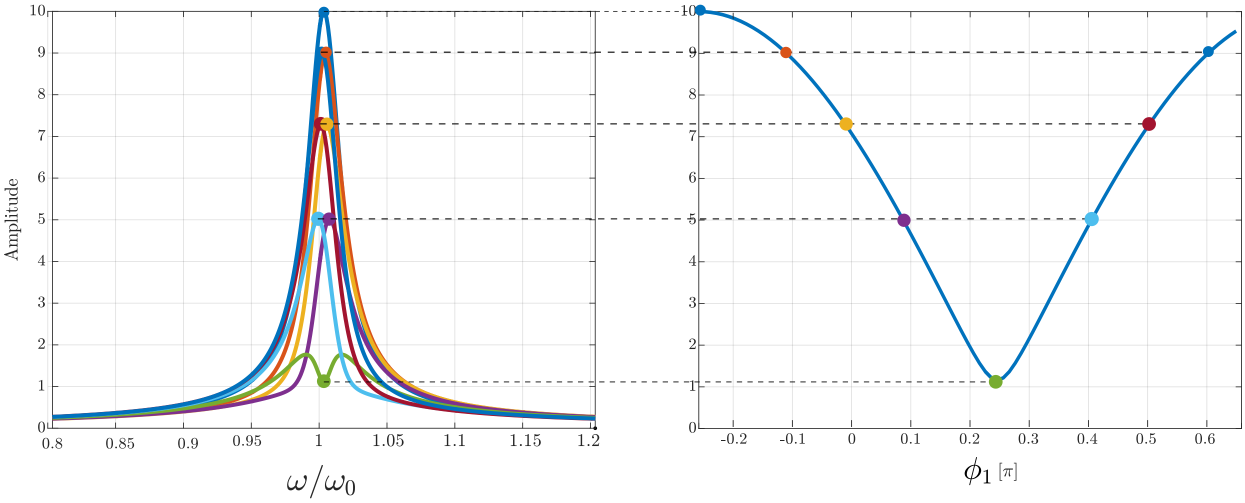

Figure 1: Constant forcing only. Left plot: amplitude of the variable part of the solution in Eq.(3) for different value of the phase between the p-th and $2p$-th harmonic of the stiffness. Right plot: Amplitudes at the resonance as a function of the phase difference, an amplification takes place at $\phi_1 = -\frac{\pi}{4}$ and an attenuation at $\phi_1 = \frac{\pi}{4}$. In this example $p = 1$ and $\phi_2 = 0$ (the second harmonic of the stiffness is the origin for the phase)

Constant and Harmonic forcing

Here, we consider that the system is statically and harmonically forced ($h_0 \neq 0$ and $\widehat{h}_n \neq 0$) around its resonance ($n\Omega \approx \omega_0$) and that $\widehat{\chi}_p \neq 0$ and $\widehat{\chi}_{2p} \neq 0$. Due to the linearity of Eq.(1), the solution to this case can be obtained by summing the results of Eq.(2) and Eq.(3). This leads to interactions between the harmonics of the force and/or the stiffness resulting in parametric amplification or attenuation effect, depending on the relative phase between $\widehat{h}_n$, $\widehat{\chi_n}$ and $\widehat{\chi}_{2n}$, that will be described in the presentation.

Conclusions

In this study, we have shown that parametric amplification in Mathieu's oscillators is possible for in the case of an harmonic forcing as well as in the case of a pure constant forcing. The parametric amplification is due to interaction between harmonics of the stiffness and/or of the forcing. This can be interesting for various fields of applications where the modification of the phases between the harmonics of the stiffness may cause a reduction in vibration levels. In particular, we think that this phenomenon might be of interest for understanding and/or designing geared systems.

References

[1] N.Sainte-Marie. A transmission-error-based gear dynamic model: Applications to single-and multi-mesh transmissions. PhD thesis, INSA Lyon, 2016.

[2] L.Ruby. Applications of the mathieu equation. American Journal of Physics, 1996.

[3] A. Nayfeh and D. Mook. Nonlinear oscillations. John wiley & sons, 1997.

[4] D.Rugar and P.Grutter. Mechanical parametric amplification and thermomechanical noise squeezing. Physical Review Letters, 1991.

[5] O.Thomas, F.Mathieu, W.Mansfield, C.Huang, S.Trolier-McKinstry, and L.Nicu. Efficient parametric amplification in micro-resonators with integrated piezoelectric actuation and sensing capabilities. Applied Physics Letters, 2013.