Effect of dry friction on a parametrically excited nonlinear oscillator

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Summary. This study proposes to investigate the effects of dry friction on the behaviour of a parametrically excited nonlinear oscillator using a pendulum as example. A harmonic balance method and time integration simulations are used to respectively compute and validate the solutions of the problem and their stability. The effects of dry friction on the behaviour of the system are discussed.

Context

The parametric resonance phenomenon comes from the excitation of a dynamical system through the modulation of one of its parameter and can be simply described by the Mathieu's equation [1]. This phenomenon has been widely used to enhance the dynamical behaviour of systems for energy harvesting or parametric amplification [2, 3]. Usually it occurs for a forcing frequency different from the resonance of the excited system and results in infinite amplitude of oscillation if no non linearities are present. The latter are thus necessary to stabilize the system motion and obtain a finite amplitude of oscillation. An example of such a system is given by the Mathieu-Duffing's equation. When adding viscous damping, a forcing amplitude threshold depending on the damping coefficient must be overcome to initiate parametric oscillations. Although dry friction is also a common source of damping, few studies deal with this kind of problems [4]. The present study proposes to investigate the effects of dry friction on the behaviour of a parametrically excited nonlinear oscillator. The governing equation is given and the example of the pendulum is used as illustration. A harmonic balance method is used to compute theoretical solutions of the problem. Time integration simulations are compared to these solutions to validate the model and its stability. The effects of dry friction on the behaviour of the pendulum and on its forcing amplitude threshold are discussed.

Theoretical motion of a parametric nonlinear oscillator with dry friction

Governing equation

The Mathieu-Duffing's equation including viscous and dry friction damping terms reads

$$\ddot{\theta} + \mu_1 \dot{\theta} + f_0(\dot{\theta}) + (\omega_0^2 - \delta\, 4\Omega^2 \cos(2\Omega t))\, \theta - \gamma\, \theta^3 = 0. \tag{1}$$

This equation is the governing equation of a pendulum parametrically excited with a vertical displacement. In this case, θ is the angular displacement of the pendulum and $\dot{\bullet}$ denotes a derivative relative to time t. $\omega_0^2 = g/l$ is the resonance angular frequency of the system with g the gravitational acceleration and l the length of the pendulum. δ and 2Ω are respectively the amplitude and the angular frequency of the forcing. A factor 2 is joined to Ω since the parametric resonance occurs at half of the excitation frequency in the case of the pendulum. The factor $4\Omega^2$ appears due to the double derivative relative to time of the forcing displacement term. The nonlinear coefficient γ comes from the linearisation of the $\sin$ term describing the motion of the pendulum. The viscous damping coefficient is μ_1 and the dry friction term is described by the non-smooth function $f_0(\dot{\theta}) = \mu_0 sign(\dot{\theta})$ if $\dot{\theta} \neq 0$ and $f_0(\dot{\theta}) \in [-\mu_0, \mu_0]$ if $\dot{\theta} = 0$, with μ_0 the dry friction coefficient.

Harmonic balance approach

The harmonic balance method (HBM) is used to find the solutions of Equation (1) using a Fourier series expansion of the angular displacement with only one harmonic :

$$\theta(t) = a(t)\ \cos(\Omega t + \beta(t)) \tag{2}$$

Substituting (2) in (1), considering the expansion of the dry friction function as a one term Fourier series and equating each harmonics in Ω and 3Ω with zero results in a system of four equations named $\mathcal{S}$. Equating all time derivatives of $\mathcal{S}$ with zero and neglecting harmonics higher than the first order, one founds that the amplitude of fixed points can be obtained solving the following equation :

$$\frac{9}{16}\,\gamma^2\, a^6 - \frac{3}{2}\,\gamma(\omega_0^2 - \Omega^2)\, a^4 + \left[(\omega_0^2 - \Omega^2)^2 + \Omega^2\mu_1^2 - 4\Omega^4\delta^2\right]\ a^2 + \frac{8}{\pi}\,\Omega\mu_0\mu_1\, a + \frac{16}{\pi^2}\,\mu_0^2 = 0 \tag{3}$$

The amplitude a is numerically computed to find the non trivial solutions of the pendulum motion. Then, the phase β of the angular displacement is computed such as

$$\tan(2\beta) = \frac{\left(\Omega\mu_1\, a + \frac{4}{\pi}\,\mu_0\right)}{\left(\frac{3}{4}\,\gamma\, a^3 - (\omega_0^2 - \Omega^2)\, a\right)} \tag{4}$$

It is worth mentioning that the trivial solutions $a = 0$ cannot be found with Equation (3) even if these solutions obviously exist according to experiments. The stability of solutions is computed using the method of varying amplitude [5]. The Jacobian $\mathcal{J}$ of the system $\mathcal{S}$ is first calculated. Then, the stability of the solutions is evaluated using the sign of its trace and determinant. Assuming $\mu_1 > 0$, the trace of $\mathcal{J}$ is found to be negative. Thus, the condition for stability is achieved when the determinant of $\mathcal{J}$ is positive.

Time integration approach

Time integration simulations of Equation (1) are used to (i) validate the solutions found using the HBM, (ii) study the trivial solutions of the system when $\mu_0 \neq 0$ and (iii) validate the solution stability computed with the method of varying amplitude. Simulations are computed using the ode45 solver from Matlab (MathWorks, Natick, USA). To avoid numerical issues due to the discontinuities brought by the dry friction term, the switch model proposed in [6] is used. Thus, the dry friction function is not regularized but replaced by a function including a transition phase in addition to the usual stick and slip phases. Backward and forward frequency sweeps are done to obtain the trivial and periodic solutions.

Results

Figure 1 presents solutions of Equation (1) computed with the HBM and time integration simulations. The black dotted line is the backbone curve of the system computed using the HBM. Red lines are the well-known solutions of the parametric pendulum without dry friction whose behaviour is softening. The trivial solution is unstable between the two bifurcation points of the periodic orbits. In this region, the pendulum necessarily jumps on the stable non trivial solution branch when the forcing amplitude is larger than the critical value given by the HBM $\delta_{cr} = \mu_1/2\Omega$. Blue lines are the solutions of the parametric pendulum with dry friction. In this case, non trivial solutions are disconnected from the trivial ones which are always stable according to time integration simulations. According to HBM, the birth of the resulting isola does not occur from a zero amplitude but from a point represented by the blue cross in Figure 1. It is worth mentioning that the HBM does not predict this birth point onto but nearby the backbone curve. The critical forcing amplitude needed to give birth to the isola depends on both the viscous and dry friction coefficient. However, the HBM does not allow the computation of an analytical value for this threshold.

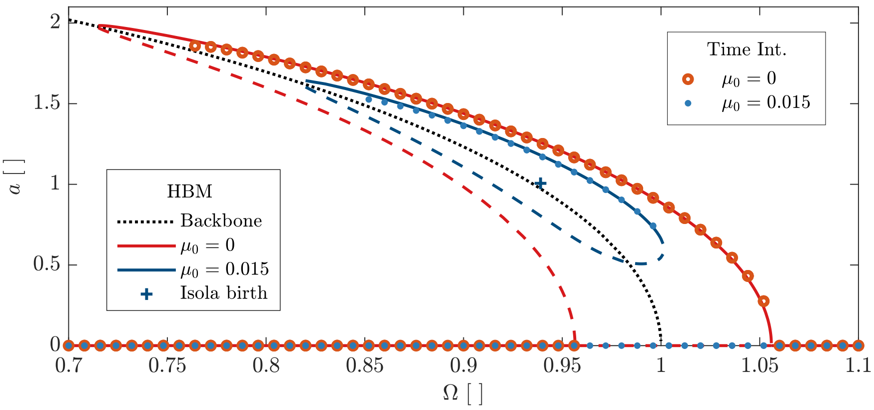

Figure 1: Solutions of Equation (1) computed with the harmonic balance method (lines) and time integration simulations (circle markers) for $\omega_0 = 1$, $\gamma = 1/6$, $\delta = 0.07$, $\mu_1 = 0.1$ and $\mu_0 = 0$ (red) or $\mu_0 = 0.015$ (blue). Plain and dashed lines correspond respectively to stable and unstable solutions. The black dotted line is the conservative solution of the system ($\delta = 0$, $\mu_1 = 0$, $\mu_0 = 0$). The blue cross corresponds to the birth of the isola.

Conclusion and perspectives

A Mathieu-Duffing's equation including a dry friction term was investigated to describe the dynamical behaviour of a parametrically excited oscillator. The example of the pendulum was used to illustrate this problem. Solutions and their stability were respectively computed using a harmonic balance approach and the method of varying amplitude. Results were validated using time integration simulations. The dry friction gave birth to isolated solutions. To further investigate these solutions, the energy principle method detailed in [7] will be used to analytically derive the critical forcing amplitude needed to give birth to the isola. The influence of other parameters like initial conditions on the existence of non trivial solutions will be also discussed.

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