Normal form based nonlinear modes: identification, experimental continuation and internal resonances applied to the acoustics of chinese gongs

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Summary. This article presents several topics related to the use of nonlinear modes and normal forms to analyse and model the vibratory response of geometrically nonlinear structures. It is first shown that the normal form theory provides a mathematically rigorous and clear framework to exhibit the mathematical form of very reduced order models. Then, this theory is applied to model the nonlinear vibrations of chinese opera gongs, that exhibits particular frequency glides in normal playing conditions, under an impulse forcing at center. It is shown that at low amplitude, a single Duffing like oscillator is sufficient to precisely explain this behaviour, due to the hardening/softening behaviour of the fundamental axisymmetric vibration mode. At larger amplitude, mode coupling are experimentally observed, well recovered by a reduced order model reduced to a few nonlinear modes involved in a 1:2:2 internal resonance.

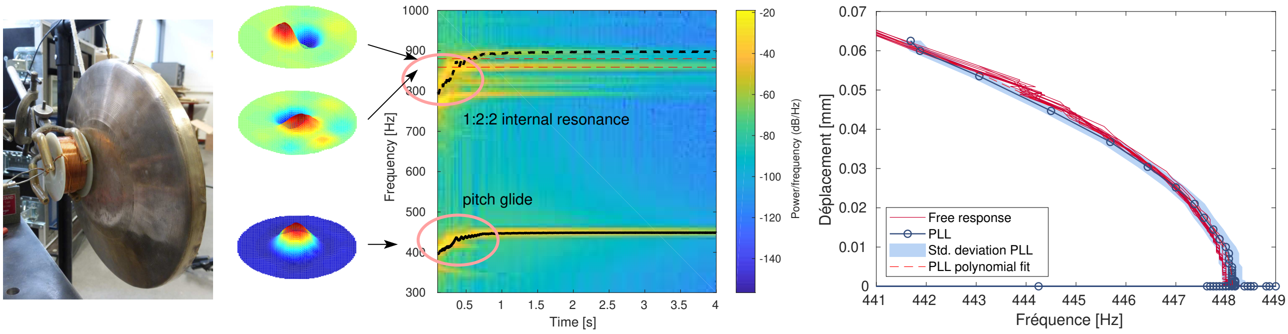

Figure 1: (left) Photograph of a chinese opera gong and the coil magnet driving system used for experimental identification; (middle) spectrogram of the acceleration signal after a mallet strike, measured by an accelerometer glued on the gong; (right) backbone curves of the fundamental nonlinear mode obtained from the free vibration regime after a mallet strike and from an experimental continuation with a phase locked loop.

Chinese opera gongs take the form of an axisymmetric thin shell, such the one shown in Fig. 1(left). When the gong is hit with a mallet at its center, a very characteristic pitch glide can be heared. This can be related to a change of the instantaneous frequency of vibrations, that can be seen on the vibration spectrogram of Fig. 1(middle) around the natural frequency of the fundamental axisymmetric mode. Then, the same spectrogram also shows that the second harmonics of the vibration signals mixes with a couple of higher modes with companion asymmetric mode shapes. In this article, we show that those two features (the pitch glide and the mode interaction) can be quantitatively recovered by reduced order models, that take the form of a few coupled nonlinear oscillators, justified by the normal form theory applied to a generic nonlinear modal model of the system.

Nonlinear modes and normal form

We consider an elastic structure whose displacement $w(\boldsymbol{x}, t)$ at time t and position $\boldsymbol{x}$ is expanded on a family of N eigenmodes of the linearized and undamped model:

$$w(\boldsymbol{x}, t) = \sum_{k=1}^{N} \Phi_k(\boldsymbol{x}) q_k(t), \tag{1}$$

where $(\omega_k, \Phi_k(\boldsymbol{x}))$ are the k-th natural angular frequency and mode shape. In free undamped vibrations, the modal coordinates $q_k(t)$ satisfy the following set of coupled nonlinear equations, for all $k = 1, \dots N$:

$$\ddot{q}_k + \omega_k^2 q_k + \sum_{i,j=1}^{N} \beta_{ij}^k q_i q_j + \sum_{i,j,l=1}^{N} \gamma_{ijl}^k q_i q_j q_k = 0, \tag{2}$$

where β_{ij}^k and γ_{ijl}^k are nonlinear coefficients stemming from the geometrical nonlinearities. Using normal forms, as introduced in [9], a nonlinear polynomial change of coordinate is introduced, leading to replace model (2) by a new one, function of the new (normal) coordinates $u_k(t)$. This new dynamical system has an important property: it involves only resonant nonlinear terms. This property enables a rigorous and straightforward truncation strategy, divided in two cases.

If there are no internal resonance relation between the oscillations frequencies, the resonant terms are only of cubic order and they do not break the invariance of the oscillators. Consequently, a motion on a single oscillator (the i-th.) is possible and takes the form:

$$u_k = 0 \ \ \forall k \neq i, \quad \ddot{u}_i + \omega_i^2 u_i + \Gamma_1 u_i^3 + \Gamma_2 u_i \dot{u}_i^2 = 0, \tag{3}$$

where (Γ_1, Γ_2) are two coefficients depending on the nonlinear coefficients β_{ij}^k and γ_{ijl}^k, that take into account the influence of all linear modes. This particular motion, linked to its invariance property, defines a nonlinear mode, whose dynamics is governed by the single oscillator (3). The values of Γ_1 and Γ_2 defines the hardening or softening feature of the nonlinear mode [9, 8].

If there is an internal resonance between some modes, the corresponding oscillators have to be kept in the dynamics and are coupled by particular nonlinear terms. There form can be easily deduced from the internal resonance relation. For instance, in the case of a 1:2 internal resonance between modes 1 and 2, there natural frequencies verify the relation $\omega_2 \simeq 2\omega_1$ and the normal form reduced order model is:

$$u_k = 0, \ \ \forall k \neq 1, 2, \quad \left\{ \begin{array}{l} \ddot{u}_1 + \omega_1^2 u_1 + \alpha_1 u_1 u_2 = 0, \\ \ddot{u}_2 + \omega_2^2 u_2 + \alpha_2 u_1^2 = 0, \end{array} \right. \tag{4}$$

where (α_1, α_2) are the coefficients of the two quadratic resonant terms [6].

If the damping is small, it's influence on the invariant manifolds geometry can be neglected and a modal viscous damping terms of the form $2\xi_k \omega_k \dot{u}_k$, with $\xi_k \ll 1$, can be added in the above models with no loss of accuracy.

Identification

If an accurate model is at hand, the values of the coefficients of the normal forms (3), (4) can be obtained from the ones of the modal model (2) using the formula of [9] or directly from a finite-element model [5]. Another strategy is to rely on experiments to identify those coefficients. At low amplitude and without internal resonance, we can show that the influence of coefficients (Γ_1, Γ_2) on the dynamics can be embedded into a single cubic coefficient, whose sign governs the hardening / softening behaviour of the nonlinear mode. As shown in [1, 3], this coefficient can be efficiently estimated with an experimental backbone curve, that can be measured by experimental continuation based on a Phase-Locked Loop (PLL). As a consequence, it is here proven that an accurate reduced order model for this low amplitude single nonlinear mode motion is a classical Duffing oscillator. This identification procedure can be extended to measure more complex dynamics, such those involving 1:1 internal resonance [2]. In the case of the internal resonance of Eq. (4), coefficients (α_1, α_2) can be estimated by experimental forced responses, as explained in [7, 4].

Acoustics of a chinese gong

The above method can be applied to chinese opera gongs, in order to investigate and explain their particular sound. Considering first their pitch glide, it is possible to extract from a free response in normal playing conditions the relation between the instantaneous frequency of oscillations as a function of the amplitude. It can be shown that it matches exactly the experimental backbone curve (Fig. 1(right)), leading to the conclusion that the characteristic pitch glide of the Chinese opera gongs is an acoustic manifestation of the hardening / softening behaviour of their fundamental nonlinear mode, whose frequency changes as a function of the amplitude because of the geometrical nonlinearities. Then, using a model of the form (4) with an additional cubic term, the 1:2:2 internal resonance of Fig. 1(middle) can also be recovered by proper estimation of coefficients (α_1, α_2).

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