Resonance analysis for a nonhomogeneous wave equation with a time-dependent coefficient in the Robin boundary condition

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Summary. This work is devoted to how the frequency of an external force effects the resonances in a one-dimensional initial-boundary value problem for a nonhomogeneous wave equation involving a Robin type of boundary condition with a time-dependent coefficient. By setting the frequency of the external force equal to ω, and the time-dependent boundary coefficient in the boundary condition equal to $k(t)$, different kinds of resonances can be obtained by numerical simulations. Next, by using the method of d'Alembert and wave reflections, we can calculate the solution $u(x,t)$ by dividing the time domain into finite intervals of length 2. Finally, the resonance results can be analyzed by the map of the solution from $t = 2n$ to $t = 2(n+1)$, which are in agreement with those obtained by using a numerical method.

Statement of the problem

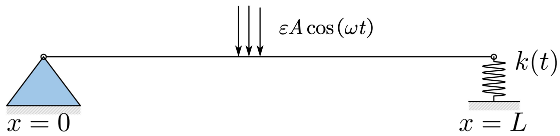

Figure 1: The transverse vibrating string with a time-varying spring-stiffness support at x=L.

In this paper we study resonance for a nonhomogeneous wave equation (see Figure 1), where one end is attached to a spring for which the stiffness properties change in time (due to fatigue, temperature change, and so on). By using Hamilton's principle, the system can be written as:

$$\rho u_{tt}(x,t) - P u_{xx}(x,t) = \varepsilon cos(\omega t), \quad 0 < x < L, \quad t > 0, \tag{1}$$

where ρ is the mass density, P is the axial tension which is assumed to be constant, L is the distance between the supports, and u describes the lateral displacement of the string. $\varepsilon cos(\omega t)$ is an external force acting on the whole string, where ε and ω are constants. The boundary conditions are:

$$u(0,t) = 0, \quad P u_x(L,t) + k(t)u(L,t) = 0, \quad t > 0, \tag{2}$$

where $k(t)$ is the time-varying stiffness of the spring at $x = L$. The boundary condition at $x = 0$ is a Dirichlet type of boundary condition, and the boundary condition at $x = L$ is a Robin type of boundary condition with a time-dependent coefficient $k(t)$. Based on the Buckingham Pi theorem,the governing equation (1), the boundary conditions (2), and the initial conditions can be transformed to the following non-dimensional form:

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = \varepsilon cos(\omega t), \quad 0 < x < 1, \quad t > 0, \\ u(0,t) = 0, \quad u_x(1,t) + k(t)u(1,t) = 0, \quad t > 0, \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 < x < 1. \end{cases} \tag{3}$$

Numerical example of Resonance

This section is devoted to presenting some numerical simulations on the dynamical resonance behavior of system (3) for two cases. Let us first assume that $\varepsilon = 1$ and the following initial conditions are given:

$$f(x) = sin^2(1.7155x), \quad g(x) = 0, \quad 0 < x < 1. \tag{4}$$

Case 1: k(t) is constant

Choosing $k(t) = 2$ and setting $\omega = \lambda_1$ (λ_i satisfies the transcendental equation $-\frac{1}{k}\lambda_i = tan(\lambda_i)$) the solution of the nonhomogeneous wave equation (3) can be obtained, and is given as in Figure 2-1 (where a resonance arises).

Case 2: k(t) is not constant ($k(t) = \frac{1}{at+1}$)

Let $k(t) = \frac{1}{at+1}$. We know the eigenvalues λ satisfy approximately $-(at+1)\lambda = tan(\lambda)$ (see Figure 3). By giving fixed values for "ω" and "a", different solution shapes can be obtained as time increases (see Figure 2-2 to 2-4). Resonances might occur (or not) depending on the choices for "ω" and "a".

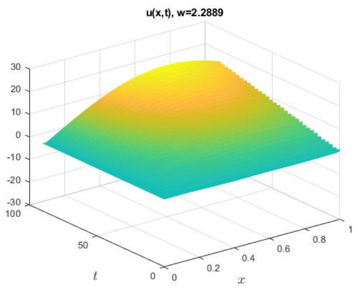

Figure 2-1: The solution $u(x,t)$ with k=2, ω=2.2889.

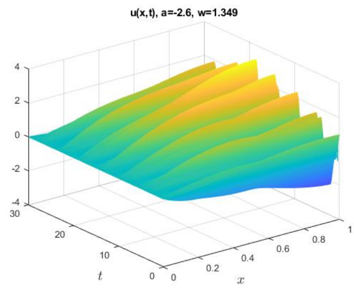

Figure 2-2: $\omega = 1.329$, resonance zone is around (2,16).

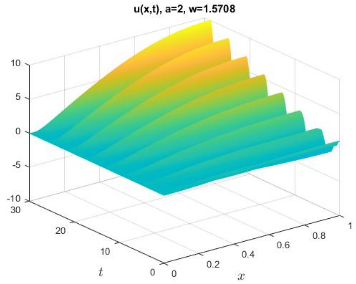

Figure 2-3: Resonance arises when $\omega = \frac{\pi}{2}$.

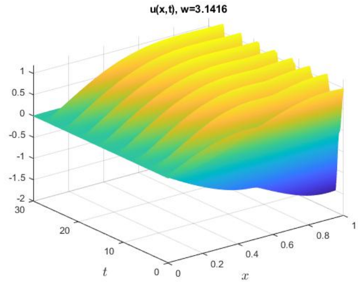

Figure 2-4: No resonance when $\omega = \pi$.

Resonance (unbounded solution) analysis

The analytical solution based on the method of d'Alembert

According to the method of d'Alembert (see [1]), the general solution to Eq.(3) is given by

$$u(x,t) = \frac{1}{2}[f(x-t) + f(x+t)] + \frac{1}{2}\int_{x-t}^{x+t} g(s)ds + \varepsilon \int_0^t (t-\tau)cos(\omega\tau)d\tau.$$

It should be noted that the functions f and g are only defined on the interval [0,1]. To extend f and g on the whole domain $(-\infty, +\infty)$, the boundary conditions should be considered.

The nonhomogeneous wave equation we considered above in Eq.(3) has a propagation speed of 1, which implies that the vibration information at the point $x = x_i$ and $t = 0$ will propagate into two different directions with speed 1, and the information will be back to the position x_i at $t = 2$ as shown Figure 4-1. Furthermore, Figure 4-2 shows the domain of dependence. Then by treating the state at $t = 2$ as a new initial condition and using the same extension procedures, the information needed to calculate the solution of the equation up to every time can be obtained (for details see [2]).

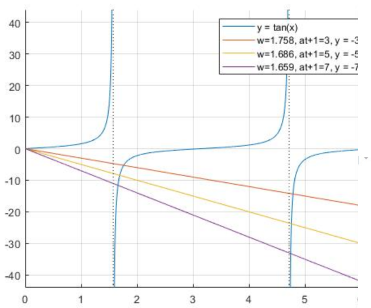

Figure 3: The resonance produced by different values of ω.

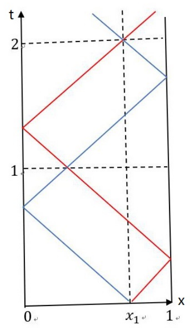

Figure 4-1: Wave reflections.

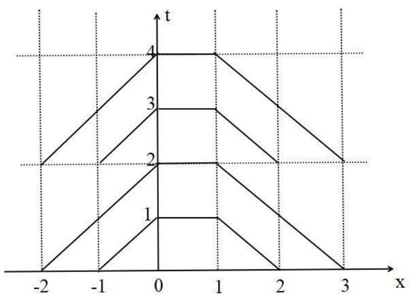

Figure 4-2: Domain of dependence.

Solution and mapping

The resonance results can be analyzed by the map of the solution based on the proposed method (the method of d'Alembert), from $t = 2n$ to $t = 2(n+1)$, which turns out to be in complete agreement with those obtained by using a numerical method.

References

[1] J. d'Alembert, (1747) Researches sur la courbe que forme une corde tendue mise en vibrations (Researches on the curve that a tense cord forms when set into vibration). *Hist. l'Académie R. Des Sci. Belles Lettres*, **3**:214-249.

[2] Wim T. van Horssen, Yandong Wang, Guohua Cao, (2018) On solving wave equations on fixed bounded intervals involving Robin boundary conditions with time-dependent coefficients. *Journal of Sound and Vibration*, **424**:263-271.