A Partitioned Finite Element Method (PFEM) for power-preserving discretization of port-Hamiltonian systems (pHs) with polynomial nonlinearity

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Summary. The Partitioned Finite Element Method introduced in [7] provides a structure-preserving discretization for the solution of systems of boundary controlled and observed Partial Differential Equations (PDEs), formulated as distributed-parameter port-Hamiltonian systems (pHs). In particular, the energy balance is preserved at the discrete level. This method, already well-developped for linear systems, is also suitable for nonlinear systems with polynomial nonlinearity, such as the 2D Shallow Water Equations, or the full von-Kármán plate equations.

Port-Hamiltonian systems (pHs)

These are dynamical systems ruled by a Hamiltonian function and conservation laws, together with interaction ports for control $\boldsymbol{u}$ through actuators, and observation or measurements $\boldsymbol{y}$ via sensors; this modelling tool proves very useful for the analysis and control of multiphysics systems: see e.g. [8] for a general presentation. PHs can be finite dimensional (i.e. described by Ordinary Differential Equations (ODEs) with a finite number of d.o.f), or infinite dimensional (i.e. described by Partial Differential Equations PDEs), see [13]. In both cases, all the variables involved in the description do have a clear physical meaning, in contrast with many methods available in the mathematical literature: in particular, anisotropic and heterogeneous media can be accounted for in a very natural way, with no extra complication w.r.t the isotropic and homogeneous cases.

General setting [3]

Let $\mathcal{H}$ the Hamiltonian functional, and (α_1, α_2) the energy variables in the domain Ω. The co-energy variables are defined as the variational derivatives of the Hamiltonian w.r.t. these energy variables: $e_i := \delta_{\alpha_i}\mathcal{H}$. The dynamical system reads:

$$\partial_t \begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = \begin{bmatrix}0 & -\mathcal{L}^*\\ \mathcal{L} & 0\end{bmatrix}\begin{pmatrix}e_1\\ e_2\end{pmatrix}, \qquad \begin{aligned} u_\partial &= \gamma_0 e_1,\\ y_\partial &= \gamma_\perp e_2, \end{aligned} \tag{1}$$

and the differential operator $\mathcal{L}$ and its adjoint $\mathcal{L}^*$ satisfy the following abstract Green's formula:

$$\langle e_2,\, \mathcal{L}\, e_1\rangle_{L^2(\Omega,\mathbb{A}_2)} - \langle \mathcal{L}^*\, e_2,\, e_1\rangle_{L^2(\Omega,\mathbb{A}_1)} = \langle \gamma_0 e_1,\, \gamma_\perp e_2\rangle_{\partial\Omega}\,. \tag{2}$$

The energy variables $\alpha_i \in L^2(\Omega, \mathbb{A}_i)$, where the sets $\mathbb{A}_i$ are either scalar, vectorial or tensorial quantities. The co-energy variables e_i belong to some appropriate Sobolev spaces, namely: $e_1 \in H^{\mathcal{L}} := \left\{v_1 \in L^2(\Omega, \mathbb{A}_1) |\ \mathcal{L}v_1 \in L^2(\Omega, \mathbb{A}_2)\right\}$, and $e_2 \in H^{\mathcal{L}^*} := \left\{v_2 \in L^2(\Omega, \mathbb{A}_2) |\ \mathcal{L}^* v_2 \in L^2(\Omega, \mathbb{A}_1)\right\}$. Then, the evolution of the Hamiltonian is given by:

$$\frac{d}{dt}\mathcal{H}(\alpha_1(t), \alpha_2(t)) = \langle u_\partial(t),\, y_\partial(t)\rangle_{\partial\Omega}\ , \tag{3}$$

corresponding to a lossless open system, a generalization of a conservative closed system.

Worked-out examples in structural, fluid mechanics and electromagnetism

The practical cases dealt with so far are:

- $\mathcal{L} = \mathrm{div}$ and $\mathcal{L}^* = -\,\mathbf{grad}$, for vectorial and scalar fields in 2D in [12] and [5],
- $\mathcal{L} = \mathrm{Div}$ and $\mathcal{L}^* = -\,\mathrm{Grad}$, for tensorial and vectorial fields in 2D in [1] and [4],
- $\mathcal{L} = \mathbf{curl}$ and $\mathcal{L}^* = \mathbf{curl}$, for vectorial fields in 3D in [9],
- $\mathcal{L} = \mathrm{div}\,\mathrm{Div}$ and $\mathcal{L}^* = \mathrm{Grad}\,\mathbf{grad} = \mathrm{Hess}$ for tensorial and scalar fields in 2D in [2].

In each case, the energy and co-energy variables are defined in accordance with continuum mechanics and physics.

The Partitioned Finite Element Method (PFEM)

This method makes use of the finite element method to open systems of conservation laws, i.e. with collocated boundary controls and observations; it was first introduced in [6], fully detailed in [7], and extended to damped systems in [10].

Principle

The Partitioned Finite Element Method boils down to 3 steps: first provide a weak formulation of the coupled first order system, second perfom an integration by parts on one of the two lines as to highlight the desired boundary control, and third apply a Galerkin procedure by choosing finite element families for each component including the boundary, giving rise to large sparse matrices.

Let us denote the shape functions φ_i^1 for the first variables, φ_j^2 for the second variable and ψ_k for the boundary variables, set the vectors $\Phi_1 := [\varphi_1^1 \cdots \varphi_{N_1}^1]$, $\Phi_2 := [\varphi_1^2 \cdots \varphi_{N_2}^2]$ and $\Psi := [\psi_1 \cdots \psi_{N_\partial}]$, and define the discrete Hamiltonian as $H(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) := \mathcal{H}(\Phi_1 \, \boldsymbol{\alpha}_1, \Phi_2 \, \boldsymbol{\alpha}_2)$, we end up with the following finite-dimensional port-Hamiltonian system:

$$\begin{aligned} \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \frac{d}{dt} \begin{pmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \end{pmatrix} &= \begin{bmatrix} 0 & -L^T \\ L & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{pmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \boldsymbol{u}_\partial, \qquad & M_1 \boldsymbol{e}_1 := \nabla_{\boldsymbol{\alpha}_1} H, \\ & & M_2 \boldsymbol{e}_2 := \nabla_{\boldsymbol{\alpha}_2} H, \\ M_\partial \, \boldsymbol{y}_\partial &= \begin{bmatrix} 0 & B_2^T \end{bmatrix} \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{pmatrix}. \end{aligned} \tag{4}$$

The mass matrices M_i of dimension $N_i \times N_i$ are symmetric and positive definite. Matrix L is $N_2 \times N_1$, and the boundary control matrix B_2 is $N_2 \times N_\partial$. Then, mimicking (3), the power balance for open systems is preserved at the discrete level:

$$\frac{d}{dt} H(\boldsymbol{\alpha}_1(t), \boldsymbol{\alpha}_2(t)) = \boldsymbol{u}_\partial^T M_\partial \boldsymbol{y}_\partial \, . \tag{5}$$

Linear examples

When the Hamiltonian is a quadratic and separable functional of the energy variables, the co-energy variables are linear w.r.t. the energy variables, and an explicit closed-form for (4) can be easily obtained, either in terms of $\boldsymbol{\alpha}_i$, or of $\boldsymbol{e}_i$. This strategy has been fully developped and worked out on the anisotropic heterogeneous 2D wave equation in [12], on the Reissner-Mindlin thick plate in [1], on the Kirchhoff-Love thin plate in [2] and also on Maxwell's equations in 3D in [9].

Nonlinear examples

Now the same strategy applies to nonlinear models, keeping the energy and co-energy variables apart. In this case though, the link between the co-energy variable $\boldsymbol{e}_i$ and the energy variables $\boldsymbol{\alpha}_i$ is no more linear and requires some special care. However, in the case of a polynomial nonlinearity, explicit relations can be provided at the discrete level, and more important, these relations can be computed off-line (i.e. an extra application of the FEM does not prove necessary at each time step). This has been worked out on the irrotational 2D Shallow Water in [5] with a nonlinear and non-separable Hamiltonian, on the heat equation with internal energy as Hamiltonian in [11]. The full von-Kármán plate model in [4] is another candidate.

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