Nonlinear dynamics of topological lattices

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Summary. Due to the recent discovery of topological insulators in condensed matter physics, a new notion of topology has emerged in association with the intrinsic wave dispersion of a structure. It has led to several mechanical designs with robust localization of energy in space—potentially offering novel applications in energy harvesting, vibration isolation, and structure health monitoring. The framework of topology fundamentally rests on the linear dynamics of the system. In this work, we present our recent efforts to understand the interplay between nonlinearity and topology in mechanical systems. Our system obeys the dynamics that is governed by a second-order differential equation akin to electronic circuits. In particular, we study one-dimensional nonlinear lattices of both Fermi–Pasta–Ulam–Tsingou and Klein-Gordon types and discuss linear stability of topological states, soliton formation, and nonlinearity-induced topological transition. The findings highlight the effect of nonlinearity on the characteristics of topologically-robust edge states and the role of topology in interpreting purely nonlinear states.

Introduction

It is well-known that a periodic arrangement of structural constituents or properties possesses band gaps, and thus, can filter a range of frequencies from propagating in the medium. Recently, the topological characterization of such systems has enabled researchers to see them in a new light. Two periodic systems showing similar band gaps can still be different on topological grounds. This difference is quantified in terms of the topological invariant calculated from band dispersion. At the physical level, the difference manifests as robust edge/surface states in topologically-nontrivial systems. Such modes are "topologically protected" as small impurities in the system do not affect their presence [1].
The majority of studies on the topological mechanical lattices have so far relied on linear wave dynamics. Some open questions in the area are: What is the effect of nonlinearity on the characteristics of topologically-robust edge states? How can the topological framework be used as a novel tool to interpret purely nonlinear states? Along these lines, in this work, using analytical and numerical tools, we examine three particular aspects of the interplay between topology and nonlinearity in mechanical systems shown next.

Linear stability of topological states

As shown in Fig. 1(a), we take a 1D chain that consists of equal masses and two interconnecting linear springs of stiffness coefficients $1+\gamma$ and $1-\gamma$. Each mass is grounded with a nonlinear spring of cubic nonlinearity, such that the force (F)-deformation(x) profile is $F = \gamma_0 x + \Gamma x^3$, where γ_0 is the linearized ground stiffness and Γ is the parameter for nonlinearity. In the linear limit ($\Gamma = 0$), this lattice possesses a band gap due to nonzero γ. Depending on the sign of γ, the system makes a topological transition from a trivial to nontrivial state [2]. Consequently, the topologically-nontrivial system supports a robust edge state with frequency inside the band gap. We examine how the onsite nonlinearity affects the form and stability of such a state. In Figs. 1(b) and 1(c), we show the nonlinear edge states (obtained through the continuation of the linear edge state) and their long-time dynamics for the chain with *softening* ($\Gamma < 0$) and *stiffening* ($\Gamma > 0$) types of nonlinearity, respectively. We find that the topological edge states are generally unstable due to oscillatory instabilities, leading to the delocalization of energy into the bulk of the chain. However, we find a frequency regime in the case of the *stiffening* nonlinearity that can support linearly stable edge states. This could potentially be useful in confining high-amplitude vibrations at the edges of structures for a long duration.

Moving Dirac soliton excites topological states

In Fig. 1(d), we take another chain similar to previous chain in the linearized limit; however, it has cubic nonlinearity in interconnecting springs. The force (F)-deformation(dx) profile of two alternating springs is $F = (1-\gamma)dx + \Gamma dx^3$ and $F = (1+\gamma)dx + \Gamma dx^3$. Depending on the sign of γ, the system again supports a topological edge state in the linear limit. The frequency of the state lies inside the band gap. Due to nonlinearity, it is possible to find spatially localized solutions that lie in the bulk of the periodic lattice with their frequency inside the band gap. Therefore, for a small band gap ($\gamma << 1$) and nonlinearity ($\Gamma << 1$), we employ the continuum approximation and find Dirac solitons residing inside the band gap. A stationary soliton is shown in Fig. 1(e). We also find a family of moving solitons in the lattice. We demonstrate numerically in Fig. 1(f) that such moving solitons can be used to excite edge states in lattices from far distances without actually placing the vibration source on the edge of the lattice.

Self-induced topological state by nonlinearity management

To achieve a topological transition in a mechanical lattice using nonlinearity, we take a 1D chain that consists of equal masses and two types of nonlinear springs, a *stiffening* and a *softening* spring, periodically alternating along the chain as shown in Fig. 1(g). We choose force (F)-deformation (dx) profiles for two springs as $F = (1-\gamma)dx + \Gamma dx^3$ and $F = (1+\gamma)dx - \Gamma dx^3$. In the linear limit ($\Gamma = 0$) the system supports a band gap; however, the system is in a

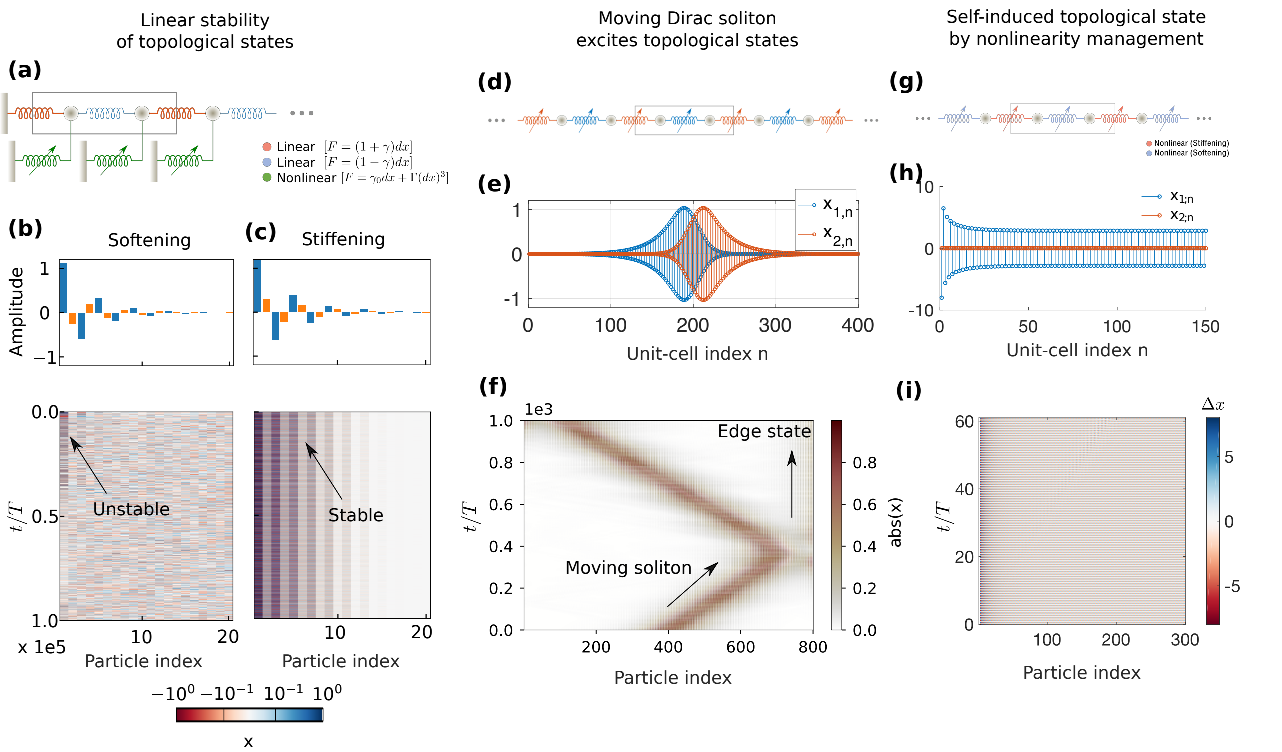

Figure 1: (a) A 1D chain with two linear springs alternating along the length and onsite nonlinearity. (b) One of the nonlinear edge states for *softening* nonlinearity. Below is its spatiotemporal diagram verifying instability. (c) One of the nonlinear edge states for *stiffening* nonlinearity. Below is its spatiotemporal diagram verifying linear stability. (d) A 1D chain with two nonlinear springs alternating along the length. (e) Dirac soliton residing inside the band gap. (f) Spatiotemporal diagram showing a moving Dirac soliton the excites an edge state localized on the right boundary of the lattice. (g) A 1D chain with two types of nonlinear springs (one *stiffening* and another *softening*) alternating along the length. (h) Analytically-obtained edge state in the lattice as a result of the amplitude-dependent topological transition. (i) Spatiotemporal diagram verifying the existence of such a solution in the lattice.

topologically trivial state with no state localized on the edges of the chain. For a system with a small band gap ($\gamma << 1$) and nonlinearity ($\Gamma << 1$), when the amplitude is increased, we witness an effective closure and reopening of the band gap. This leads to an amplitude-dependent topological transition, and we observe the emergence of a self-induced edge state in the system. We show an analytically obtained profile in Fig. 1(h). We note that this has a unique profile different from the ones seen in linear systems. Instead of the amplitude decaying to zero farther away from the edge, we note that it saturates to a nonzero amplitude [3]. We then verify the existence of this solution by performing full numerical simulations for the lattice. We take the nonlinear solution shown in Fig. 1(h) as initial conditions and obtain the spatiotemporal strain diagram in Figs. 1(i). We do not observe any significant scattering for a short time duration, which verifies the existence of the edge state in the system. This study, therefore, highlights novel ways to tailor nonlinearity in mechanical lattices to achieve a topological transition and localize spontaneously some energy at the edges of the system.

Conclusions

In this work, we discuss three different aspects of studying the nonlinear dynamics of topological lattices. First, we find that nonlinearity can generally make a topological edge state linearly unstable. However, depending on the details of nonlinearity, it is possible to find high-amplitude edge states that are linearly stable. Second, nonlinearity makes it possible to achieve spatial localization of energy in defect-free lattices. These solutions, e.g., moving soliton in our case, can be used to excite topological edge states in the lattices from farther distances. And third, an amplitude-dependent topological transition can be achieved by nonlinearity management in mechanical lattices, such that exotic nonlinear states emerge spontaneously on the edges.

References

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