Solitary wave-like solutions in hyperelastic tubes conveying inviscid and viscous fluid

Vasily Vedeneev*

Lomonosov Moscow State University, Moscow, Russia

Summary. We study possible steady states of an infinitely long tube made of a hyperelastic membrane and conveying either an inviscid, or a viscous fluid with power-law rheology. The tube model is geometrically and physically nonlinear; the fluid model is limited to smooth changes in the tube's radius. For the inviscid case, we analyse the tube's stretch and flow velocity range at which standing solitary waves of both the swelling and the necking type exist. For the viscous case, we show that a steady-state solution exists only for sufficiently small flow speeds and that it has a form of a kink wave; solitary waves do not exist. For the case of a semi-infinite tube (infinite either upstream or downstream), there exist both kink and solitary wave solutions. For finite-length tubes, there exist solutions of any kind, i.e. in the form of pieces of kink waves, solitary waves, and periodic waves.

Introduction

Nonlinear waves in fluid-filled elastic tubes play an important role in problems of the cardiovascular system [1,7]. Solitary wave solutions are used for the analysis of pulse waves as well as for the study of the formation of aneurysms [3,4]. As a rule, such solutions are theoretically analysed without consideration of the fluid viscosity [2,5]. The goal of this study is to include the viscosity into account and to study its effect on solitary-wave-like solutions.

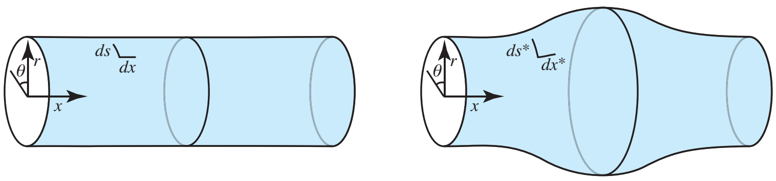

Figure 1: Cylindrical membrane tube in the initial and the deformed state.

Problem formulation and results

Governing equations

We consider a cylindrical membrane tube with a circular cross-section with a thickness of h and a radius of R, made of incompressible hyperelastic Gent material (Fig. 1). The ratio h/R is sufficiently small for the bending stresses to be neglected compared to the membrane stresses. The tube conveys a non-Newtonian viscous fluid whose rheology obeys a power law. We restrict ourselves to axisymmetric motion. In [2], a self-contained derivation of the exact equations of motion for the case of an inviscid fluid was given. We re-derive these equations and upgrade it by the viscous fluid forces taken into account. The fluid equations are considered in the formulation [9, 10]. With these assumptions, closed system of four differential equations is derived.

Analysis: inviscid fluid

If the fluid viscosity is neglected and the velocity is constant at each cross-section, the system of equation has four first integrals, as shown in [2]. We prove that the existence of these integrals retains if the velocity distribution is non-constant, but the viscosity is still neglected. The system of first integrals is reduced to a two-dimensional dynamic system, which is analysed by its phase plane (Fig. 2).

It is known [6] that for a quiescent fluid (or, equivalently, if a constant pressure is set in the tube) in a tube that is axially unstretched at infinity (axial stretch $\lambda_{10} = 1$), a standing solitary wave in the form of a localised swelling exists for a range of far-field circumferential stretches $1.18 < \lambda_{20} < 1.69$. We show that in the case of a moving fluid, there is a range of velocities ($0.063 \leq v_{f0} \leq 0.58$ for far-field radial stretch $\lambda_{20} = 1.5$), in which there exists, simultaneously with

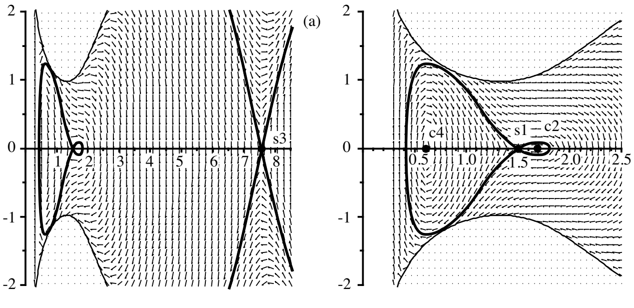

Figure 2: Vector field and separatrices of the stationary saddle points for $\lambda_{10} = 1$, $\lambda_{20} = 1.5$, and $v_{f0} = 0.4$. General view (a), enlarged view in the area of the separatrix loops (b).

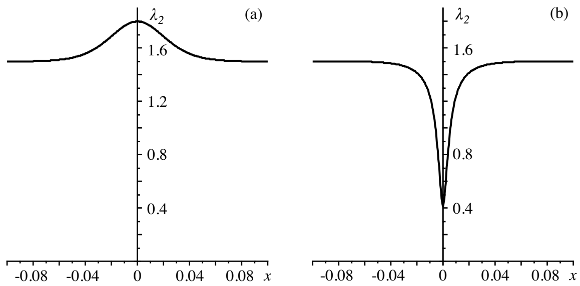

Figure 3: Swelling (a) and necking (b) solitary waves $\lambda_2(x)$ for $\lambda_{10} = 1$, $\lambda_{20} = 1.5$, and $v_{f0} = 0.4$.

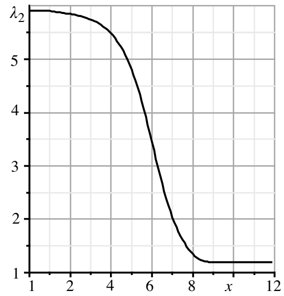

Figure 4: Kink-like solution $\lambda_2(x)$ for infinite-length tube conveying viscous fluid.

the standing swelling solitary wave, a standing necking solitary wave. At a lower fluid velocity, there is only a swelling solitary wave; for larger velocities, no solitary waves exist. Note that in a model of a geometrically and physically linear tube, in which only the nonlinearity of the flow was taken into account [8], there always exists, for any nonzero flow velocity, only a standing necking solitary wave. Thus, both the existence of a standing swelling solitary wave and the limited range of fluid velocities for which a standing necking solitary wave exists are consequences of the physical and geometrical nonlinearities of the tube model.

Analysis: viscous fluid

When a viscous fluid moves, only two of the four first integrals exist, and the motion of the trajectory on the phase plane is accompanied by a simultaneous change of the vector field. First, we prove that there are limit stretch states of the tube as $x \to -\infty$ and $x \to +\infty$, with the stretches λ_1 and λ_2 tending to constants but the stresses tending to infinities to compensate for the fluid pressure and the longitudinal stress caused by the fluid viscosity, which are infinitely growing upstream and infinitely decreasing downstream. The transition between these limit states occurs in the central section of the tube and exists only if the fluid velocity is sufficiently small. In this case there is a unique solution linking the states at infinity in the form of a monotonic decrease in the radius downstream, i.e. a kink-like solution. Localised swelling or necking solutions for a tube that is infinitely long in both directions do not exist. However, such solutions exist if the tube is infinitely long in only one direction, either downstream or upstream. But solutions in which a semi-infinite tube has multiple neckings or swellings do not exist. For finite-length tubes, there exist 'pieces' of both swelling and necking solitary waves, as well as close-to-solitary-wave solutions with a finite number of successive swellings or neckings.

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References

[1] Cao Y., Zheng Y., Li G-Y., Jiang Y. (2019) Elastodiagnosis of diseases: A review. *Extreme Mech. Letters* **27**:102–123.

[2] Epstein M., Johnston C.R. (2001) On the exact speed and amplitude of solitary waves in fluid-filled elastic tubes. *Proc. R. Soc. Lond. A* **457**:1195–1213.

[3] Dehghani H., Desena-Galarza D., Jha N.K., Reinoso J., Merodio J. (2019) Bifurcation and post-bifurcation of an inflated and extended residually-stressed circular cylindrical tube with application to aneurysms initiation and propagation in arterial wall tissue. *Finite Elem. Anal. Des.* **161**:51–60.

[4] de Gelidia S., Bucchia A. (2019) Comparative finite element modelling of aneurysm formation and physiologic inflation in the descending aorta. *Comp. Meth. Biomech. Biomed. Eng.* **22(15)**:1197–1208.

[5] Il'ichev A.T., Shargatov V.A., Fu Y.B. (2020) Characterization and dynamical stability of fully nonlinear strain solitary waves in a fluid-filled hyperelastic membrane tube. *Acta Mech.* **231**:4095–4110.

[6] Pearce S.P., Fu Y.B. (2010) Characterization and stability of localized bulging/necking in inflated membrane tubes. *IMA J. Appl. Math.* **75**:581–602.

[7] Pedley T.J. (2003) Arterial and Venous Fluid Dynamics. In Cardiovascular fluid mechanics (ed. G. Pedrizzetti and K. Perktold), chap. 1. Springer.

[8] Poroshina A.B., Vedeneev V.V. (2018) Existence and uniqueness of steady state of elastic tubes conveying power law fluid. *Russian Journal of Biomechanics* **22(2)**:169–193.

[9] Vedeneev V.V., Poroshina A.B. (2018) Stability of an elastic tube conveying a non-Newtonian fluid and having a locally weakened section. *Proc. Steklov Inst. Math* **300**:34–55.

[10] Yushutin V.S. (2012) Stability of flow of a nonlinear viscous power-law hardening medium in a deformable channel. *Moscow Univ. Mech. Bull.* **67 (4)**:99–102.