Multiscale analysis for traveling-pulse solutions to the stochastic FitzHugh-Nagumo equations

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Summary. We consider the stochastic FitzHugh-Nagumo equations, whose deterministic equivalent allows for fast and stable traveling-pulse solutions. In this talk, we investigate the stability of fast pulses in case of additive noise and derive a multiscale decomposition for small stochastic forcing. Our method is based on adapting the wave velocity by solving a stochastic ordinary differential equation and tracking perturbations of the wave meeting a stochastic partial differential equation coupled to an ordinary differential equation. Previous works have focused on applying this method to scalar equations, such as the stochastic Nagumo equation, which carry a self-adjoint structure. This structure is lost in case of the FitzHugh-Nagumo system and the linearization does not generate an analytic semigroup. We show that this problem can be overcome by making use of Riesz spectral projections in a certain way. This provides a relevant generalization as our approach appears to be applicable also to general stochastic nerve-axon equations, the stochastic periodically-forced NLS equation, or systems of stochastic reaction-diffusion equations with spectrum parallel to the imaginary axis.

The talk and the following presentation are based on and adapted from [4], respectively.

The stochastic FitzHugh-Nagumo equations

Consider the *stochastic FitzHugh-Nagumo equations* driven by additive noise

$$\mathrm{d}u(t,x) = \left(\nu\partial_x^2 u(t,x) + f(u(t,x)) - v(t,x)\right)\mathrm{d}t + \sigma\,\mathrm{d}W(t,x) \quad \text{for} \quad (t,x)\in\mathbb{R}_+\times\mathbb{R}, \tag{1a}$$

$$\mathrm{d}v(t,x) = \varepsilon\,(u(t,x) - \gamma v(t,x))\,\mathrm{d}t \quad \text{for} \quad (t,x)\in\mathbb{R}_+\times\mathbb{R}, \tag{1b}$$

in which the *electric potential* u and the *gating variable* v are functions of time t and position x on a neural axon. Here, $\nu\partial_x^2 u$ determines the spatial diffusion on the axon with $\nu > 0$ and the reaction term $f(u)$ is nonlinear and typically reads $f(u) = \chi(u)\,u\,(1-u)\,(u-a)$ with $0 < a < 1$ and a suitable cut off χ. The term $\sigma\,\mathrm{d}W$ is additive noise, with W denoting an infinite-dimensional Wiener process taking values in a suitable Hilbert space H. The parameter $\varepsilon > 0$ determines the coupling strength of the electric potential u to the gating variable v and will be assumed to be sufficiently small. The parameter $\gamma > 0$ is a decay constant of the gating variable v.

Traveling-pulse solutions and their stability

The deterministic variant of (1) with $\sigma = 0$ allows for traveling-pulse solutions of the form $u(t,x) = \hat{u}(\xi)$ and $v(t,x) = \hat{v}(\xi)$, where $\xi = x + st$ and $s \in \mathbb{R}$ is the pulse's velocity. The vectorial function $\left(\hat{u}, \frac{\mathrm{d}\hat{u}}{\mathrm{d}\xi}, \hat{v}\right)$ is a nontrivial homoclinic orbit of a three-dimensional continuous and autonomous dynamical system

$$\frac{\mathrm{d}\hat{u}}{\mathrm{d}\xi} = \hat{u}', \quad \frac{\mathrm{d}\hat{u}'}{\mathrm{d}\xi} = \frac{1}{\nu}\left(s\hat{u}' - f(\hat{u}) + \hat{v}\right), \quad \frac{\mathrm{d}\hat{v}}{\mathrm{d}\xi} = \frac{\varepsilon}{s}\left(\hat{u} - \gamma\hat{v}\right) \quad \text{for} \quad \xi\in\mathbb{R}. \tag{2}$$

Existence of nontrivial pulses with $(\hat{u},\hat{v})^{\mathrm{t}} \to (0,0)^{\mathrm{t}}$ as $\xi \to \pm\infty$ is ensured if $\gamma \geq 0$ is sufficiently small [11]. Indeed, a singular perturbation argument around $\varepsilon = 0$ and $s = 0$ yields existence of a *slow pulse* with wave speed $s \approx 0$, which by Sturm-Liouville theorey turns out to be unstable. Nonetheless, there is also a *fast pulse*, in what follows denoted by $(\hat{u},\hat{v})$, corresponding to much higher wave speeds $s = s(f,\varepsilon)$, constructed in [1, 2] employing the method of isolating blocks of the fast and slow subsystems [3]. Using an Evans function analysis, stability of the fast pulse in the space of bounded uniformly continuous functions was first proved in [7]. Stability in $L^2(\mathbb{R};\mathbb{R}^2)$ and $H^1(\mathbb{R};\mathbb{R}^2)$ was proved in [5, 12, 13].

Our approach for computing the velocity correction

We investigate the following decomposition

$$u(t,x) = \hat{u}\left(x + st + \varphi(t)\right) + u_\varphi(t,x), \qquad v(t,x) = \hat{v}\left(x + st + \varphi(t)\right) + v_\varphi(t,x) \tag{3}$$

of solutions to (1), in which the function $\varphi(t)$ is a random correction to the pulse's position, and $u_\varphi(t,x)$ and $v_\varphi(t,x)$ denote lower-order fluctuations that are uniquely defined through (3) for any $\varphi = \varphi(t)$. Ideally, we would like to choose φ to minimize the distance in the direction of the traveling wave between the solution $X := (u,v)^{\mathrm{t}}$ of the FitzHugh-Nagumo SPDEs (1) and the suitably translated traveling wave $\hat{X} = (\hat{u},\hat{v})^{\mathrm{t}}$, i.e.,

$$\varphi(t) \in \operatorname{argmin}_{\varphi\in\mathbb{R}} \left\| \Pi^0_{st+\varphi}\left(X(t,\cdot) - \hat{X}(\cdot + st + \varphi)\right)\right\|_H^2, \quad \text{with} \quad X := \begin{pmatrix} u \\ v \end{pmatrix}, \quad \hat{X} := \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix}, \tag{4}$$

where $\|\cdot\|_H$ is the H-norm in the spatial variable and $\Pi^0_{st+\varphi}$ is a suitable projection onto the pulse such that $\Pi^0_{st+\varphi}\frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}(\cdot + st+\varphi) = \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}(\cdot + st+\varphi)$. However, as problem (4) is not necessarily convex, uniqueness of a minimizer is not certain. Following [8] and replacing (4) by the weaker condition for a critical point of finding $\varphi = \varphi(t)$, we may impose

$$0 = \left(\Pi^0_{st+\varphi}\left(X(t,\cdot) - \hat{X}(\cdot + st+\varphi)\right), \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}(\cdot + st+\varphi)\right)_H, \tag{5}$$

where $(\cdot,\cdot)_H$ is the inner product of H. This approach has been employed in [6] for more general classes of SPDE systems, but the results only hold up to the first stopping time when the local minimum becomes a saddle point. Here, we follow [8], i.e., $\varphi(t)$ is approximated by a process $\varphi^m(t)$ fulfilling the random ordinary differential equation

$$\frac{\mathrm{d}\varphi^m}{\mathrm{d}t}(t) = m\left(\Pi^0_{st+\varphi^m(t)}\left(X(t,\cdot) - \hat{X}(\cdot + st+\varphi^m(t))\right), \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}\left(\cdot + st+\varphi^m(t)\right)\right)_H \tag{6}$$

for given initial data and a sufficiently large relaxation parameter $m > 0$.

Results

We obtain the following results:

1. We establish existence and uniqueness of solutions to (1) using the variational approach for equations with locally monotone coefficients [9, 10].
2. We give a short proof of deterministic stability in $L^2(\mathbb{R};\mathbb{R}^2)$ of the fast pulse $\hat{X} = (\hat{u},\hat{v})^{\mathrm{t}}$, which simplifies the presentations in [5, 12, 13].
3. We derive an SODE defining the correction of the wave's velocity. The leading-order part of this SODE contains a linear damping term due to the relaxation method of the frame and additive stochastic fluctuations obtained from projecting the infinite-dimensional noise onto a translate of $\frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}$.
4. We prove a multiscale expansion
$$X(t,\cdot) = \hat{X}(\cdot + st + \sigma\varphi_0^m(t)) + \sigma X_0^m(t,\cdot) + o(\sigma), \tag{7}$$
where $(\sigma\varphi_0^m, \sigma X_0^m)$ solves the scaled linearized evolution of (φ^m, X^m).
5. We consider the limit $m \to \infty$ of immediate relaxation. Here, we prove that
   - the multiscale expansion (7) remains satisfied as $m \to \infty$,
   - as $\sigma \searrow 0$ the first exit time where the multiscale decomposition cannot be guaranteed to hold anymore converges to the entire time interval,
   - by employing deterministic stability, the second moment $\mathbb{E}\,\|X_0^\infty(t,\cdot)\|_H^2$ of fluctuations transverse to the traveling pulse mode after correcting the wave velocity stays bounded,
   - the second moment $\mathbb{E}\left\|X(t,\cdot) - \hat{X}(\cdot + st)\right\|_H^2$ of fluctuations transverse to the traveling wave mode without correcting the wave velocity asymptotically can grow linearly in time.

We conclude by discussing generalizations of our work, such as the application to other (systems of) SPDEs with spectrum parallel to the imaginary axis or the stability of more complicated patterns.

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