Resonant Triads of Acoustic-Gravity Waves in Shallow Water

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Summary. We consider the interaction of wave disturbances for shallow water of uniform depth. When we allow for compressibility effects in the linear theory, acoustic waves are assumed to be decoupled from free-surface gravity waves due to their disparity in propagation. However, it is possible to have energy exchange between acoustic and gravity wave modes given a non-linear interaction through a resonating triad. In this study, we analyse the case of a single gravity wave and two counter propagating acoustic modes which are of comparable length scales, but differing temporal scales. We derive amplitude evolution equations to describe the cyclic exchange of energy they exhibit through asymptotic methods, and implement them numerically to gain insight into the energy exchange of the triad with respect to the steepness of the waves. We find that the interaction allows a periodic exchange of energy in the triad where the steepness parameter is proportional to the a magnitude and inversely proportional to the period.

Introduction

In classical water-wave theory, the effects of compressibility are ignored, which stems from the idea that acoustic waves are essentially decoupled from free-surface waves. This assumption can be rationalised for the linear theory due to the speed of sound in water exceeding that of the maximum phase speed of surface waves, thus giving rise to differing spatial and/or temporal scales in the acoustic modes, and a somewhat decoupled system. However, when considering the non-linearity of the two types of waves, this assumption may not be as well justified. In a seminal paper by Longuet-Higgins [1], it has been shown that quadratic interactions of surface gravity waves can excite compression modes when we consider water of finite depth. It was argued that the formulation of oceanic microseisms could be due to such nonlinear coupling. Through studying quadratic interactions of two counter propagating wave trains of the same frequency, the behaviour was not in keeping with the classical theory, due to a non-decaying pressure component. Accounting for compressibility gave rise to resonance in the second order when an acoustic mode had double the frequency of the surface wave. Thus, the effects of compressibility must be considered to allow for this non-decaying component.

In this work we consider a non-linear coupling of acoustic and gravity waves in water of finite depth, and their interaction through a resonating triad. In recent years, numerical evidence has been presented that the resonant behaviour seen in Longeut-Higgins (1950) is a particular example of a resonant triad for two surface waves travelling in opposing directions and a propagating acoustic wave mode[2]. Also shown was that the amplitude of the free acoustic-gravity wave which was generated at resonance, or near-resonance conditions, was significantly larger than that of the bound acoustic-gravity wave, being generated far from resonance conditions. It was argued that the resonating triad amplitudes should be governed by a system of ordinary differential equations, and evolve in time, being governed by amplitude evolution equations of the same form as that of a standard resonant triad, see [3]. Following this, asymptotic theory was developed for resonant triad interactions concerning acoustic-gravity waves, taking waves of comparable temporal but differing spatial scales[4]. As Kadri and Stiassnie (2013)[2] suggested, it was shown that there is a resonance interaction between a given triad of two opposing gravity waves and a long-crested acoustic wave. Due to the interaction time scale being longer than that of a standard resonant triad, cubic terms were needed to display the wave self-interaction. What followed later was the derivation of evolution equations through asymptotic methods for both the acoustic wave and both gravity waves[4].

In the following we proceed with a similar approach as Kadri and Akylas, (2016)[4], but consider a triad of two counter-propagating acoustic modes, and a single gravity wave. In light of this, we consider differing temporal, but comparable spatial scales of the acoustic-gravity modes, and use multiple scale analysis to derive the evolution equations for the given amplitudes of the waves. Section 2 details the formulation of the the problem, deriving the governing equations and finding the dispersion relations through linearisation. Section 3 discusses the resonance conditions and the cyclic exchange of energy. Section 4 derivation for the evolution equations for a normal mode wave form, and Section 5 presents a numerical solution to the problem with results and conclusions on the interaction and its corresponding energy exchange.

Preliminaries

We look at the propagation of surface and acoustic wave disturbances in water of a constant depth h with a rigid bottom ($z = -h$), accounting for the effects of compressibility and gravity. In this setting we consider water as an inviscid barotropic fluid such that the density ρ of the fluid is solely a function of its pressure p, the motion of such fluid to be irrotational, and the speed of sound $c = (dp/d\rho)^{1/2}$ to be constant. An important parameter μ, defined as

$$\mu^2 = \frac{gh}{c^2}, \tag{1}$$

for g the gravitational acceleration, controls the effects of compressibility relative to gravity. Generally we can say that $\mu \ll 1$ due to the fact that the speed of sound in water, $c = 1.5 \times 10^3\,\mathrm{m\,s^{-1}}$, and would surpass the maximal phase velocity of gravity waves $(gh)^{1/2}$.

The following analysis is focused on nonlinear interactions of a single gravity wave mode with two acoustic wave modes, all of similar spatial but differing temporal scales. The temporal disparity arises from the frequency of the gravity wave being much smaller than that of the acoustic waves. To interpret μ, we can take $\tau_{ac} \sim h/c \implies \tau_{gr} \sim (\lambda/g)^{1/2}$, where τ_{ac} is the acoustic time scale and τ_{gr} the gravity time scale. If we take the scales to be comparable, this implies that $\tau_{ac} \sim \mu\tau_{gr}$, where μ can be thought of as the ratio of temporal scales. Thus, to introduce non-dimensional variables, we can take the time scale and length scale as $(h/g)^{1/2}$ and h respectively.

Due to the assumption of an irrotational fluid, the problem can be formulated in terms of a velocity potential $\phi(x, z, t)$ such that we have a velocity field $\boldsymbol{u} = \boldsymbol{\nabla}\phi$. The governing equation for $\phi(x, z, t)$ within the fluid interior is determined by combining continuity with the unsteady Bernoulli equation[1, 4], such that $\phi(x, z, t)$ satisfies

$$\frac{1}{\mu^2}(\phi_{xx} + \phi_{zz}) - \phi_{tt} - \phi_z - |\boldsymbol{\nabla}\phi|_t^2 - \frac{1}{2}(\phi_x|\boldsymbol{\nabla}\phi|_x^2 + \phi_z|\boldsymbol{\nabla}\phi|_x^z) = 0. \tag{2}$$

The standard kinematic and dynamic conditions are applicable at the free surface $z = \eta(x, t)$, and it is sufficient to satisfy these conditions up to third order in the perturbations for the weakly nonlinear analysis that is to follow. Once the two free-surface conditions have been expanded about $z = 0$, η can be expressed in terms of ϕ at the desired order. Thus, it is possible to obtain the boundary condition for ϕ on $z = 0$ [4]

$$\begin{aligned}\phi_{tt} + \phi_z + |\boldsymbol{\nabla}\phi|_t^2 - (\phi_t(\phi_{tt} + \phi_z))_z + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{\nabla}(|\boldsymbol{\nabla}\phi|^2) - (\phi_t|\boldsymbol{\nabla}\phi|_t^2)_z \\ -\frac{1}{2}\{(\phi_{tt} + \phi_z)(|\boldsymbol{\nabla}\phi|^2 - \phi_t^2)\}_z = 0 \text{ for } (z = 0)\end{aligned} \tag{3}$$

At the rigid bottom, the standard no penetration condition is derived

$$\phi_z = 0 \qquad (z = -1) \tag{4}$$

We now analyse the linear problem which is responsible for the disparity in temporal scales. We can drop the nonlinear terms in (2), (3) and (4) and assume ϕ as

$$\phi = f(z)\exp\left(\frac{1}{2}\mu^2 z\right)\exp\left(\mathrm{i}\left(kx - \sigma t\right)\right) \tag{5}$$

being the normal mode form, and seek wave modes propagating along x with wave number and frequency k and σ respectively. By substituting (5) into (2), (3) and (4), the boundary-value problem for the vertical profile $f(z)$ becomes

$$\begin{aligned}\frac{\mathrm{d}^2\hat{f}}{\mathrm{d}z^2} + \Omega^2 f = 0,\ \text{for } (-1 < z < 0) \\ -\omega^2\hat{f} + \mu^2\frac{\mathrm{d}\hat{f}}{\mathrm{d}z} + \frac{1}{2}\mu^4\hat{f} = 0,\ \text{for } (z = 0) \\ \frac{\mathrm{d}\hat{f}}{\mathrm{d}z} + \frac{1}{2}\mu^2\hat{f} = 0,\ \text{for } (z = -1)\end{aligned} \tag{6}$$

where $\Omega^2 = \omega^2 - k^2 - \frac{\mu^4}{4}$ and $\omega = \mu\sigma$. As $\mu \ll 1$, we assume the solution of this system to be oscillatory, giving us the solution

$$\hat{f} = \cos\Omega(z+1) - \frac{\mu^2}{2\Omega}\sin\Omega(z+1). \tag{7}$$

The dispersion relation for the gravity mode can be obtained in the standard way by substitution of the solution (7) into the boundary condition, specifically at $z = 0$, such that

$$\sigma^2 = \lambda\tanh\lambda \tag{8}$$

where ignoring compressibility gives $k = \lambda$. To find a similar relation for the acoustic mode, we first notice that $\Omega^2 > 0$, implying that we can take $\Omega = \sqrt{\omega^2 - k^2} + O(\mu^4)$. Thus, by substituting our general solution back into the boundary condition at $z = 0$ as well as our expression for Ω, we have the acoustic dispersion relation

$$\omega^2 = \omega_n^2 + k^2 + \frac{\omega_n^2 - k^2}{\omega_n^2 + k^2}\mu^2 + O(\mu^4), \tag{9}$$

where $\omega_n = (n + \frac{1}{2})\pi$.

Resonance Conditions

It has been seen numerically that if we allow for the influence of compressibility, two gravity waves can interact resonantly with an acoustic-gravity wave [2, 4]. In this study, we consider if the same holds true for two counter propagating acoustic modes (k_+, ω_+) and (k_-, ω_-), and a single gravity mode $(k, \mu\sigma)$. In order for these to form a resonant triad, they must satisfy the prescribed resonance conditions

$$\text{(i)} \quad k_+ + k_- = k \quad \text{and} \quad \text{(ii)} \quad \omega_+ - \omega_- = \mu\sigma. \tag{10}$$

as well as the appropriate dispersion relations (8), (9). Hence, we investigate said triads in the limit as $\mu \ll 1$.
If we define

$$\text{(I)} \quad \omega_\pm = \tilde{\omega} \pm \frac{\mu\sigma}{2} \quad \text{and} \quad \text{(II)} \quad k_\pm = \frac{k}{2} \tag{11}$$

then by (I) and our acoustic dispersion relation (9),

$$\begin{aligned} \tilde{\omega}^2 \pm \mu\sigma\tilde{\omega} + \frac{\mu^2\sigma^2}{4} &= \omega_n^2 + k_\pm^2 + \frac{\omega_n^2 - k_\pm^2}{\omega_n^2 + k_\pm^2}\mu^2 + O(\mu^4) \\ \implies k_\pm^2 &= \tilde{\omega}^2 - \omega_n^2 \pm \mu\sigma\tilde{\omega} + O(\mu^2). \end{aligned} \tag{12}$$

Thus, if these conditions are satisfied along with the given dispersion relations, then the gravity mode can form a resonant triad with two counter-propagating acoustic waves for the given conditions. Under these conditions, we expect interactions between the resonant triad to results in a somewhat cyclic exchange of energy between the participants. As is seen in [4], due to having disparity in the temporal scales, the interaction time scale and the amplitude evolution equations will vary compared with the standard theory on resonance interactions.

Amplitude Evolution Equations

We look to derive the amplitude equations for our resonant triad. The two acoustic modes of complex amplitudes $A_\pm$ with frequencies $\omega_\pm$ interact with a single gravity mode of amplitude S and frequency $\mu\sigma$ such that $A_+ = \mathrm{e}^{\mathrm{i}(k_+x-\omega_+t)}$, $A_- = \mathrm{e}^{\mathrm{i}(k_-x-\omega_-t)}$, and $S = \mathrm{e}^{\mathrm{i}(kx-\mu\sigma t)}$. Thus, under appropriate assumptions, the expanded velocity potential can be seen as

$$\begin{aligned} \varphi =& \epsilon\{A_+(X,T)\cos\omega_n(z+1)\mathrm{e}^{\mathrm{i}(k_+x-\omega_+t)} + c.c.\} \\ &+\epsilon\{A_-(X,T)\cos\omega_n(z+1)\mathrm{e}^{\mathrm{i}(k_-x+\omega_-t)} + c.c.\} \\ &+\epsilon\{S(X,T)\cosh\lambda(z+1)\mathrm{e}^{\mathrm{i}(kx-\epsilon^{\frac{1}{2}}\sigma t)} + c.c.\} \quad + \quad . \quad . \quad . \end{aligned} \tag{13}$$

where $X = \epsilon x$, $\lambda^2 = k^2 - \mu^2\sigma^2$ and $c.c$ is the complex conjugate. The amplitudes depend on the "slow" time $T = \epsilon t$, where the wave steepness $0 < \epsilon \ll 1$ is related to μ as $\mu = \epsilon^{\frac{1}{2}}$. We begin by substituting (13) into our governing equations (2) through (4), and collect terms based on those which are proportional to $\mathrm{e}^{\mathrm{i}(kx-\mu\sigma t)}$ and $\mathrm{e}^{\mathrm{i}(k_+x-\omega_+t)}$. By doing this, we are left with terms which contribute towards resonance in the form of a reduced boundary-value problem. We then formulate a solvability condition based on this, whereby the amplitude evolution equations will follow. We start with the gravity mode by collecting terms which are proportional to $\mathrm{e}^{\mathrm{i}(kx-\mu\sigma t)}$, and impose a correction to (13) of leading order $O(\epsilon^{\frac{5}{2}})$ of the form

$$\epsilon^{\frac{5}{2}}\left\{F(X,z,T)\exp\Big\{\mathrm{i}(kx-\epsilon^{\frac{1}{2}}\sigma t)\Big\} + c.c\right\}, \tag{14}$$

such that F satisfies the boundary-value problem

$$\begin{aligned} \epsilon^{\frac{5}{2}}F_{zz} - k^2\epsilon^{\frac{5}{2}}F &= R_1 \text{ for } (-1 < z < 0) \\ \epsilon^{\frac{5}{2}}F_z &= R_2 \text{ for } (z = 0) \\ F_z &= 0 \text{ for } (z = -1) \end{aligned} \tag{15}$$

where

$$\begin{aligned} R_1 =& -2\mathrm{i}\epsilon k\frac{\partial S}{\partial X}\cosh\lambda(z+1) - 2\mathrm{i}\epsilon^{\frac{5}{2}}\sigma\frac{\partial S}{\partial T}\cosh\lambda(z+1) - \epsilon^2\sigma^2 S\cosh\lambda(z+1) + \epsilon\lambda S\sinh\lambda(z+1) \\ &+ 2\mathrm{i}\epsilon^{\frac{5}{2}}k_+k_-\sigma A_+A_-\cos^2\omega_n(z+1) - 2\mathrm{i}\epsilon^{\frac{5}{2}}\omega_n^2\sigma A_+A_-\sin^2\omega_n(z+1) \\ R_2 =& 2\mathrm{i}\epsilon^{\frac{5}{2}}\sigma\frac{\partial S}{\partial T}\cosh\lambda + \epsilon^2\sigma^2 S\cosh\lambda - \epsilon\lambda S\sinh\lambda + 2\mathrm{i}\epsilon^{\frac{5}{2}}\omega_n^2\sigma A_+A_- \end{aligned} \tag{16}$$

The solution to the corresponding homogeneous system is $\cosh kz$, so we must be able to employ a certain solvability condition for the in-homogeneous problem (15). Here, we multiply both sides of the field equation by $\cosh kz$ and integrate over the whole domain $-1 < z < 0$, such that

$$kR_2 + R_1\Big|_{z=-1}\sinh k = k\int_{-1}^{0} R_1\cosh kz\,\mathrm{d}z. \tag{17}$$

Finally, we obtain the evolution equation for the gravity mode by substituting our expressions for R_1 and R_2 into the solvability condition (17), resulting in

$$\frac{\partial S}{\partial T} - \frac{k}{\epsilon^{\frac{3}{2}}}\Big(\frac{\sigma}{\lambda^2-\sigma^2}\Big)\frac{\partial S}{\partial X} = -\frac{\sigma S}{2\mathrm{i}\epsilon^{\frac{3}{2}}}\Big\{\epsilon\sigma^2 - \frac{\lambda^2}{\lambda^2-\sigma^2}\Big\} + \frac{A_+A_-}{\lambda\cosh\lambda - \sinh\lambda}\Big\{\Psi + (\sinh\lambda - k)\omega_n^2 - \frac{k^2}{4}\sinh\lambda\Big\} \quad (18)$$

where $\Psi = \frac{k^4+2k^2\omega_n^2}{4k^2+16\omega_n^2} - \frac{2\omega_n^4}{2k^2+8\omega_n^2}$. Similarly, we employ a correction of

$$\epsilon^2\big\{F(X,z,T)\exp\{\mathrm{i}(k_\pm x - \omega_\pm t)\} + c.c.\big\} \quad (19)$$

so that for leading order of $O(\epsilon^2)$, we are left with acoustic-gravity interaction and cubic acoustic self interaction terms. Here, the solution to the homogeneous system is $\cos\omega_n(z+1)$, and we can formulate the solvability condition in a similar way as above. After multiplication by the homogeneous solution, and integrating over the domain, we arrive at the condition

$$-\frac{\omega_n}{\omega_\pm^2}R_4(-1)^n = \int_{-1}^{0} R_3\cos\omega_n(z+1)\,\mathrm{d}z \quad (20)$$

where

$$\begin{aligned} R_3 =& -2\mathrm{i}\epsilon k_\pm\frac{\partial A_\pm}{\partial X}\cos\omega_n(z+1) - 2\mathrm{i}\epsilon^2\omega_\pm\frac{\partial A_\pm}{\partial T}\cos\omega_n(z+1) - \epsilon\omega_n A_\pm\sin\omega_n(z+1)\\ &+ 2\mathrm{i}\epsilon^2 k_\mp k\omega_\pm A_\mp S\cos\omega_n(z+1)\cosh\lambda(z+1)\\ &+ 2\mathrm{i}\epsilon^2\omega_n\omega_\pm\lambda A_\mp S\sin\omega_n(z+1)\sinh\lambda(z+1) = 0,\\ R_4 =& \epsilon\omega_n A_\pm(-1)^n - 2\mathrm{i}\epsilon^2\omega_n\omega_\pm\lambda A_\mp S(-1)^n\sinh\lambda = 0. \end{aligned} \quad (21)$$

Again, we compute the necessary integrals and rearrange to give us the acoustic evolution equation

$$\frac{\partial A_\pm}{\partial T} + \frac{k_\pm}{\epsilon\omega_\pm}\frac{\partial A_\pm}{\partial X} = \frac{\omega_\pm^2 - 2\omega_n^2}{2\mathrm{i}\epsilon\omega_\pm^3}A_\pm - 2\sinh\lambda\Phi A_\mp S \quad (22)$$

where

$$\Phi = \frac{4\omega_n^4\lambda + \omega_n^2\lambda^3 - \omega_\pm^2\omega_n^2\lambda}{8\omega_\pm^2\omega_n^2 + 2\omega_\pm^2\lambda^2} - \frac{3\lambda^3 - 8\omega_n^2\lambda}{32\omega_n^2 - 8\lambda^2}, \quad (23)$$

Simulation and Discussion

As we now have our evolution equations, we want to gain a qualitative understanding of the interaction they convey, and whether there is periodic exchange of energy. The equations derived were solved numerically in MATLAB by an explicit Runge–Kutta method. By making use of the acoustic and gravity dispersion relations, we are able to define the variables needed in the solution process. We can then run the analyses for various initial conditions and see the temporal evolution of the triad interaction. We note that all waves are assumed to be of Gaussian form.

Figures 1 and 2 show the interaction where we assume no spatial influences. It can be seen quite clearly in figure 1 how the generation of the gravity wave plays out, and the cyclic exchange of energy, where the change in amplitude is periodic. There is an overall increase and decrease in the amplitude of the acoustic modes which is sinusoidal. The gravity wave quickly grows and decays, transferring its energy back and forth between the two acoustic modes. Similarly in figure 2, when all waves begin with amplitude of 1, we can see how the energy is shared amongst the triad. When the amplitude of the gravity wave is high, the energy has been transferred, so that of the acoustic is lower, and vice versa. In both cases, the acoustic waves mirror each other and as such the temporal evolution has been given for a single acoustic mode.

The variable ϵ is pivotal to the way the interaction plays out. It represents the steepness of the waves, so taking ϵ higher will result in greater energy transfer to the gravity wave. In the case of of figure 1, $\epsilon \sim 0.3$, which is why the maximum amplitude of the generated gravity wave does not grow above 0.15, whereas, when taking a larger epsilon in the case of figure 2, the maximum amplitude grows above 1.

We now want to analyse the energy exchange in terms of the value of the steepness, ϵ. We again look at the case where the gravity wave is generated, relating to the initial conditions $S(0) = 0$ and $A_\pm(0) = 1$. It is justifiable to take the two acoustic modes as equal for simplicity. The other parameters such as σ and $\omega_\pm$ are prescribed by the overall set up of the problem and are related through the acoustic and gravity dispersion relations as before. Under these initial conditions, we have the conservation law

$$\int_{-\infty}^{\infty}(|S|^2 + 2|A|^2)\,\mathrm{d}X = 2\int_{-\infty}^{\infty}|S_0|^2\,\mathrm{d}X \quad (24)$$

where the initial condition $S_0 \to 0$ as $X \to \pm\infty$. In our computation we took the initial condition $S_0 = \exp\{-X^2\}$, the Gaussian wave packet. Then we can see from (24) that

$$E(T) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|S|^2\,\mathrm{d}X \quad (25)$$

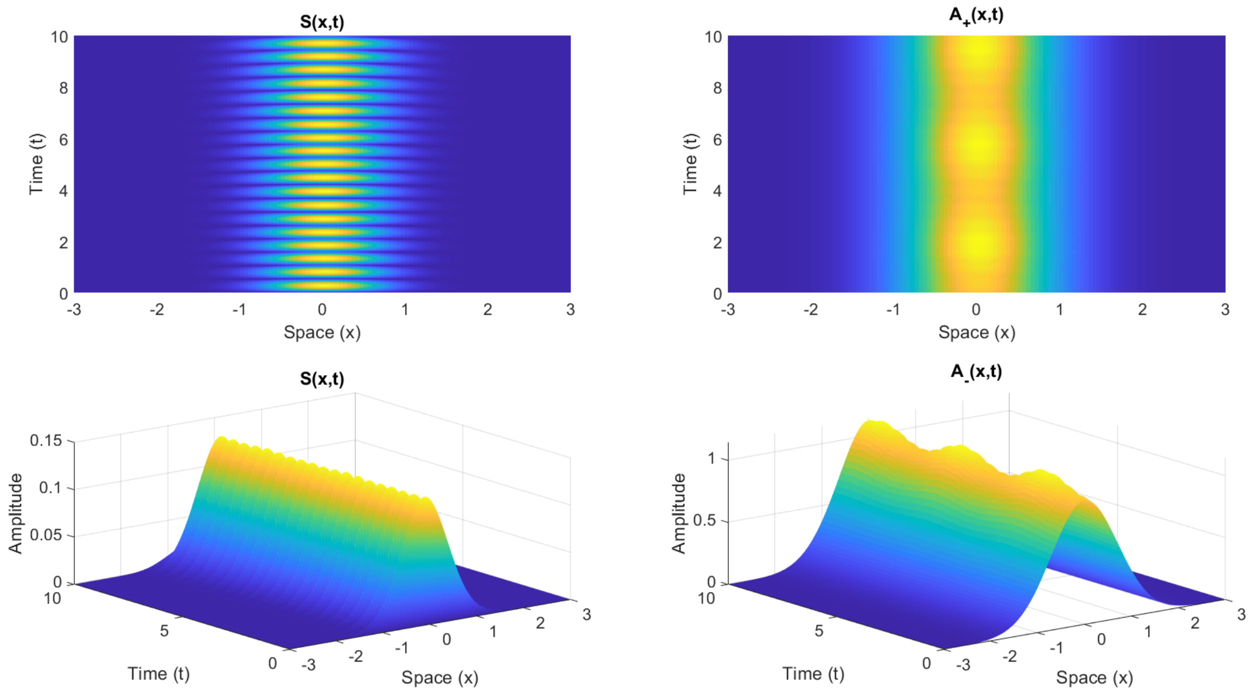

Figure 1: Generation of gravity wave with no spatial influences

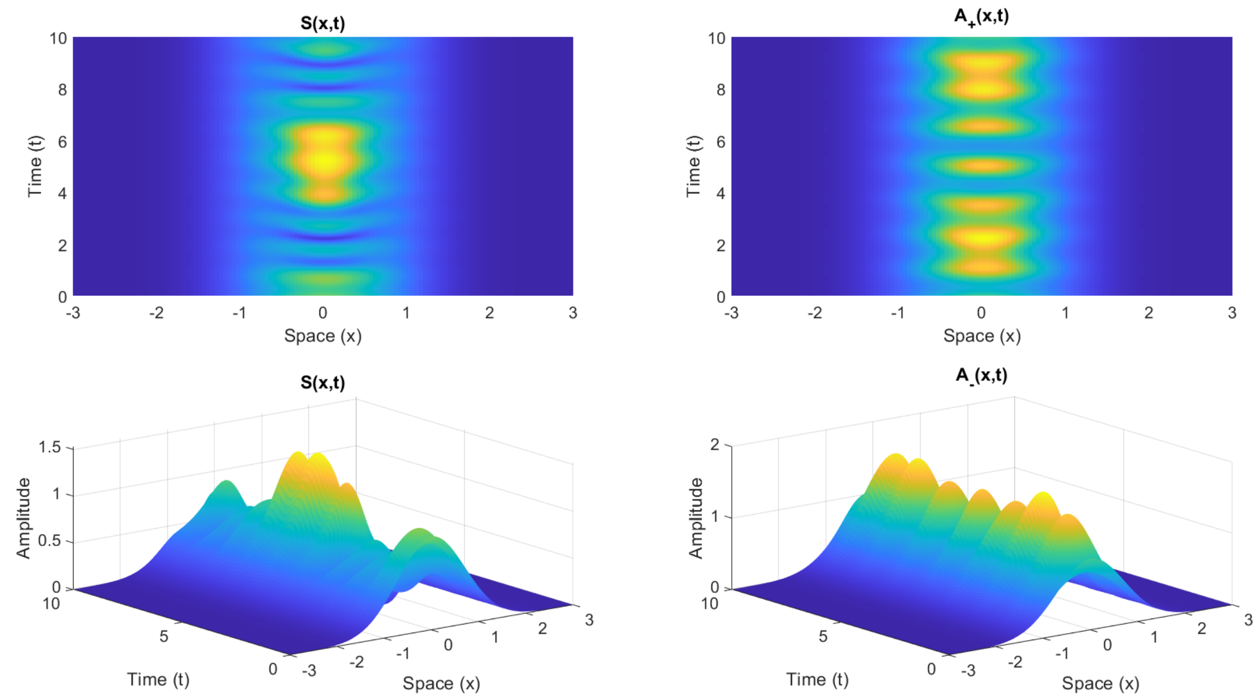

Figure 2: Interaction when all waves begin with amplitude 1 for non-spatial dependencies

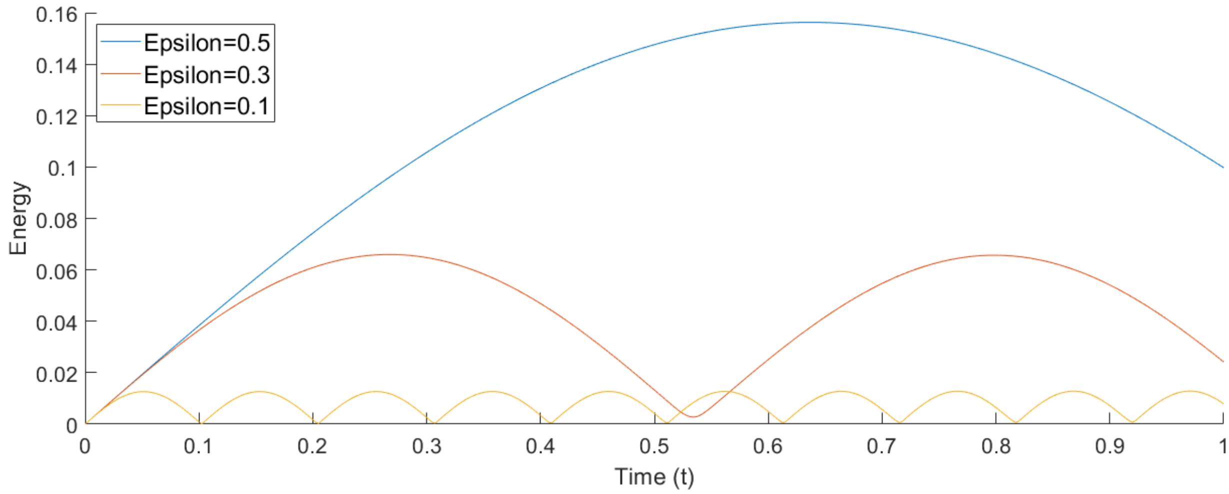

Figure 3: Transfer of energy to the gravity mode over time for varying small ϵ

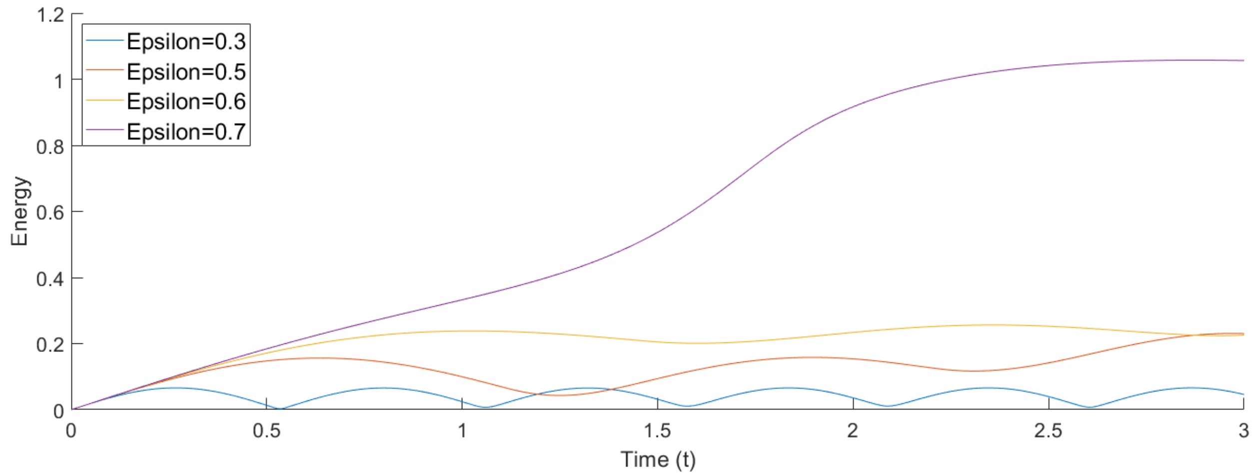

Figure 4: Transfer of energy for increasing ϵ

is the energy of the gravity wave generated. Implementing (25) numerically, we observe how increasing ϵ results in a larger amount of energy transferred to the generated gravity mode over time as shown in figure 3. Here, we used the outlined conditions above, and took small values of ϵ. As $\epsilon \to 0$, the amount of energy transferred becomes smaller, but much more frequent in its exchange, as indicated by $\epsilon = 0.1$ compared with $\epsilon = 0.5$. The periodic behaviour of curves is representative of the energy transfer between the acoustic modes and the generated gravity wave.
It is also worth analysing the rate and magnitude of energy exchange as $\epsilon \to 1$, which is shown in figure 4. As ϵ increases, we begin to have infinite periodicity occurring, where the value of the energy plateaus. This is due to our assumptions upon ϵ, mainly being that it can not be $O(1)$. So as we approach the value of 1, our resonance condition breaks down, and less energy is able to be exchanged back to the acoustic mode from the gravity wave.

Conclusion

We have derived spatial and temporal evolution equations in the case of a resonant triad of acoustic-gravity waves, and using these, have been able to analyse the energy exchange between them. Similarly to what was observed in Kadri and Akylas (2016), triad interaction of this nature differs from a standard resonant triad due to the disparity of temporal scales for the given waves. Through the presented asymptotic analysis and numerical implementation, the cyclic exchange of energy expected in a conventional resonant triad is observed, resulting from cubic self-interaction terms that were preserved when taking certain leading order. These terms directly impact the amplitude-dependent change in gravity wave frequency and the exchange of energy from surface to acoustic waves.
Similarly, the importance of the steepness parameter ϵ has been highlighted, heavily changing the interaction and effecting the cyclic exchange of energy. As the steepness increases we see larger periodicity with more energy being exchanged in each cycle. Further analysis should be conducted to investigate its effects for alternative initial conditions. Here, the generation of a single gravity wave was observed, but this choice was arbitrary. Further to this, the relationship between the interaction time and amplitudal growth would be worth investigating, as we would then have a wider picture of how

the given interaction occurs. If we were able to analyse this interaction time, and find its relationship with the amount of energy transferred, then we would have much more control over the given interaction.

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