Cantilevered Extensible Pipes Conveying Fluid: a Consistent Reduced-Order Modeling via the Extended Hamilton's Principle for Nonmaterial Volumes

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Summary. Applying the Extended Hamilton's Principle for nonmaterial volumes, a nonlinear reduced-order planar model of a cantilevered pipe conveying fluid is developed, consistently considering the effects of axial extensibility and conservation of mass associated to the internal flow. Unlike the corresponding inextensible pipe models, in which the term of transport of kinetic energy in the Extended Hamilton's Principle cancels out, in the present model such a term is not identically zero since the velocity of the flow along the pipe length is a function both of the generalized velocities and coordinates of the problem. The system dynamics is then investigated, assessing how extensibility and mass conservation affect dynamic bifurcations, by comparing root locus diagrams, and by simulating the resulting nonlinear model in some selected scenarios.

Introduction

One of the classical problems of Fluid-Structure Interactions (FSI) is the pipe conveying fluid, which is usually modeled as a flexible tube, with the use of the plug flow hypothesis for the velocity profile of the fluid. In general, the dynamic response is characterized by two types of motion depending on the internal flow velocity: stability around the static equilibrium position or dynamic Hopf bifurcations. As an open system, a proper mathematical formulation should be grounded on consistent variational principles, taking into account momentum and kinetic energy transport terms.
Benjamin [1] was responsible for the earlier investigations in this topic and considered the problem as a chain of articulated rigid pipes conveying fluid. He derived specific versions of Euler-Lagrange Equations and Hamilton's Principle for this system. Later, McIver [2] developed an extended form of Hamilton's Principle for open systems and confirmed Benjamin's work. In both studies, a term related to the transport of momentum appears.
Recently, the generalized forms of these principles were obtained by Irschik & Holl [3] and Casetta & Pesce [4]. In their derivation, an additional term related to the transport of the kinetic energy of the fluid through the nonmaterial surfaces appears, which was not present in McIver's formulation. When the inextensibility condition is assumed, the velocity of the internal plug flow relative to the pipe remains constant. Kheiri & Païdoussis [5] proved that, in this case, the term related to the transport of kinetic energy is identically zero, recovering McIver's formulation. However, when the axial extensibility of the pipe is considered, this term is not zero in the nonmaterial surface at the free end. Therefore, further discussion is needed to assess its theoretical and practical importance.
The present work proposes the formulation and analysis of a 2D reduced-order model for the cantilevered pipe conveying fluid, in which the extensibility is treated consistently, accounting for the conservation of fluid mass inside the pipe. Moreover, from Argand's type diagrams (root loci graphs) obtained through Lyapunov's indirect method, instabilities of the oscillatory modes are assessed. After these investigations, the nonlinear response in some selected scenarios is further analyzed through numerical simulations.
This introduction is followed by four sections, with the last of them bringing concluding remarks. The second section reviews the Extended Hamilton's principle for open systems. The third section describes the extensible pipe conveying fluid model, highlighting how the flow velocity expression is obtained by the application of the integral form of the conservation of mass, and deriving the equations of motion based on the definition of dimensionless variables and Galerkin's projection scheme. In the fourth section, some results are shown in the form of root loci diagrams obtained by a linearization around the static equilibrium, and on numerical integrations of the nonlinear equations of motion.

Hamilton's Principle for Open Systems or Nonmaterial Volumes

In the field of Continuum Mechanics, a material volume is understood as a volume whose closed boundary is moving with the material particles located in it. Because no transport of matter may occur, it can be also called a closed system. Whereas a nonmaterial volume - or a control volume in the terminology of Fluids Mechanics - is a fictional body which instantly coincides with a region defined by material particles; however, its control surface moves arbitrarily with respect to the material boundary, so that the particles accounted within the nonmaterial volume are not always the same, characterizing an open system. Although Hamilton's Principle and the Euler-Lagrange Equations are well formulated for material volumes or closed systems, the use of these variational approaches in problems where material transport exists is not straight-forward and needed further discussion.
According to Meirovitch [6], Hamilton's Principle for closed systems can be derived through the principle of virtual work for material volumes (D'Alembert's Principle). Assuming that δ denotes a variation in the context of Variational Calculus and $\frac{\mathrm{d}()}{\mathrm{d}t}$ is the material derivative with respect to time t, it follows,

$$\delta T + \delta W - \frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\ (\mathbf{v}\cdot\delta\mathbf{p})\ \mathrm{d}V = 0, \tag{1}$$

where V is the the material volume and $T = \frac{1}{2}\int_V \rho(\mathbf{v}\cdot\mathbf{v})\,\mathrm{d}V$ is the kinetic energy associated with the material particles of mass density ρ, position $\mathbf{p}$ with virtual displacement $\delta\mathbf{p}$ - which satisfies any imposed constraints - and velocity $\mathbf{v} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$. δW is the virtual work of the conservative and nonconservative forces.
By integrating (1) over the time interval $t_1 \leq t \leq t_2$ and considering that the configuration of the system is prescribed at the limit points (thus, $\delta\mathbf{p}(t_1) = \delta\mathbf{p}(t_2) = \mathbf{0}$), Hamilton's Principle for material volumes is obtained

$$\int_{t_1}^{t_2} [\delta T + \delta W]\,\mathrm{d}t = 0. \tag{2}$$

McIver [2] was one of the first authors who derived an extended form of Hamilton's Principle for nonmaterial volumes by using Reynolds' transport theorem in the form below, developing the third term of Equation (1) in the form,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\;(\mathbf{v}\cdot\delta\mathbf{p})\;\mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t}\int_{V_u} \rho\;(\mathbf{v}\cdot\delta\mathbf{p})\;\mathrm{d}V_u + \int_{\partial V_u} \rho\;(\mathbf{v}\cdot\delta\mathbf{p})\,(\mathbf{v}-\mathbf{u})\cdot\mathbf{n}\,\mathrm{d}\partial V_u. \tag{3}$$

Consider V_u as the the nonmaterial volume whose surface ∂V_u, with external normal unit vector $\mathbf{n}$, is moving according to a velocity field $\mathbf{u}$. Thus, McIver's statement of the principle of virtual work for open systems is obtained and the corresponding Hamilton's Principle can be derived through a similar mathematical procedure as described for material volumes,

$$\int_{t_1}^{t_2}\left[\delta T_u + \delta W - \int_{\partial V_u} \rho\;(\mathbf{v}\cdot\delta\mathbf{p})\,(\mathbf{v}-\mathbf{u})\cdot\mathbf{n}\,\mathrm{d}\partial V_u\right]\,\mathrm{d}t = 0. \tag{4}$$

McIver revealed an extra term related to the transport of momentum through the open boundary which is not present in closed systems, as observed when compared to Equation (2). Also, the author implicitly utilized $\delta T = \delta T_u$ with $\delta T_u = \frac{1}{2}\int_V \rho(\mathbf{v}\cdot\mathbf{v})\,\mathrm{d}V$ as the kinetic energy of the material particles within the nonmaterial volume V_u.
Later, it was found out that McIver's derivation does not recover the extended Euler-Lagrange equations for open systems obtained by Irschik & Holl [3]. In their paper, the authors derived the extended equations using the abstract concept of fictitious particles, which was first introduced in Truesdell & Toupin [7] for the generalization of Reynolds' transport theorem. These fictitious particles have some of same properties of their material counterparts like kinetic energy density, but their velocity is $\mathbf{u}$, the same of the control surface ∂V_u. Consider that $\mathbf{r}$ represents their position and $\delta\mathbf{r}$ is the corresponding virtual displacement. Therefore, $\mathbf{u} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$. Further investigation of Hamilton's principle for nonmaterial volumes was then necessary.
This inconsistency was addressed in a paper by Casetta & Pesce [4]. Applying the concept of fictitious particles, the following mathematical statement can be proven

$$\begin{aligned}\delta T - \frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\;(\mathbf{v}\cdot\delta\mathbf{p})\;\mathrm{d}V = \;& \delta T_u + \int_{\partial V_u} \frac{1}{2}\,\rho\,(\mathbf{v}\cdot\mathbf{v})\,(\delta\mathbf{p}-\delta\mathbf{r})\cdot\mathbf{n}\,\mathrm{d}\partial V_u \\ & -\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_u} \rho\;(\mathbf{v}\cdot\delta\mathbf{p})\;\mathrm{d}V_u - \int_{\partial V_u} \rho\;(\mathbf{v}\cdot\delta\mathbf{p})\,(\mathbf{v}-\mathbf{u})\cdot\mathbf{n}\,\mathrm{d}\partial V_u.\end{aligned} \tag{5}$$

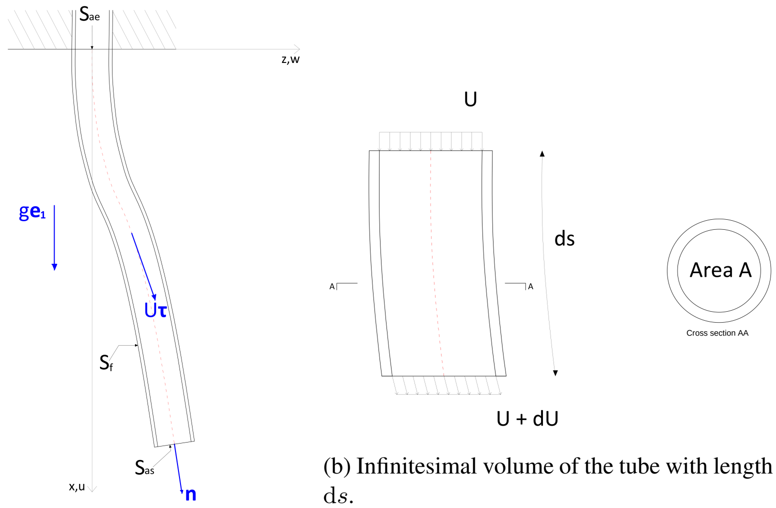
Replacing the first and the third terms of Equation (1) with Equation (5), a consistent form of the principle of virtual work for open systems is written. By integrating with respect to t over $t_1 \leq t \leq t_2$, the appropriate Hamilton's Principle was derived and Irschik & Holl [3] extended Euler-Lagrange Equations could be obtained. In this case, Hamilton's Principle for nonmaterial volumes is written as follows, [4]:

$$\int_{t_1}^{t_2}\left[\delta T_u + \delta W - \int_{\partial V_u} \rho\;(\mathbf{v}\cdot\delta\mathbf{p})\,(\mathbf{v}-\mathbf{u})\cdot\mathbf{n}\,\mathrm{d}\partial V_\mu + \int_{\partial V_u} \frac{1}{2}\,\rho\,(\mathbf{v}\cdot\mathbf{v})\,(\delta\mathbf{p}-\delta\mathbf{r})\cdot\mathbf{n}\,\mathrm{d}\partial V_u\right]\,\mathrm{d}t = 0. \tag{6}$$

In contrast to Equation (4), not only a term for the transport of momentum appears, but also one related to the flux of kinetic energy through the control surface. Respectively, the third and fourth terms.
Whenever the last term of Equation (6) is equal to zero, McIver's form - Equation (4) - is recovered, which has been extensively utilized for the modeling of the pipe conveying fluid. In fact, take a system described by a set of a finite number of generalized coordinates q_i. Assume $\dot{(\;)} = \frac{\partial(\;)}{\partial t}$. The following identities are valid: $\delta\mathbf{p} = \frac{\partial\mathbf{p}}{\partial q_i}\delta q_i = \frac{\partial\mathbf{v}}{\partial\dot{q}_i}\delta q_i$ and $\delta\mathbf{r} = \frac{\partial\mathbf{r}}{\partial q_i}\delta q_i = \frac{\partial\mathbf{u}}{\partial\dot{q}_i}\delta q_i$. In the case of an extensible pipe, from those identities, it can be proved that the last term in Eq (6) is nonzero, as outlined in the next section.

Extensible Pipe Conveying Fluid

Imagine a cantilevered, slender, cylindrical and flexible pipe constituted by an elastic linear material, that is subjected to large displacements and small strains in a 2D plane. Therefore, it can be considered as a Euler-Bernoulli beam with geometric nonlinearities. Its undeformed configuration is vertical. An internal axial, steady and incompressible flow is



(a) Extensible pipe conveying fluid.

(b) Infinitesimal volume of the tube with length $\mathrm{d}s$.

Figure 1: The 2D model.

present and its velocity profile is modeled as a plug flow. No external fluid or hydrodynamic loads exist. The pipe is considered slender, so a singular position could be described by the centerline. The system is depicted in Figure 1a.
The fixed point of the centerline is adopted as the origin of a coordinate system (x, z) with corresponding unit vectors $(\mathbf{e_1}, \mathbf{e_2})$, defining an orthonormal basis. The x-coordinate is parallel to the length of the tube in the reference position and aligned with the local gravitational field. Thus, its undeformed configuration is denoted by $(x_0, z_0) = (x_0, 0)$, with $x_0 = 0$ being the fixed end and x_0 the free end. The associated position vector is $\mathbf{R} = (x, z)$, displacements are $u = x - x_0$ and $w = z - z_0 = z$, s is the arc length coordinate along the centerline and $\kappa = \left|\frac{\partial \mathbf{R}^2}{\partial s^2}\right|$ is the curvature.
The unstretched pipe length is L, its bending stiffness is EI, axial stiffness EA, Poisson's ratio ν (volume change rate $b = 1 - 2\nu$) and linear mass m. Consider E as the Young's modulus, A as the sectional area (A_i is the internal area) and I as the area moment of inertia around the z-axis. The fluid linear mass is M and the flow has a velocity $U\boldsymbol{\tau}$, in which $\boldsymbol{\tau} = \frac{\partial \mathbf{R}}{\partial s}$ is the instantaneous tangent unit vector. The gravitational acceleration is g. The nonmaterial volume is constituted by the open surfaces $S_{ae} \cup S_{as}$ and the closed boundary S_f.

On the modeling of the internal flow velocity

If the pipe is considered inextensible, the velocity field U is equal to the constant U_0. When the extensibility condition is utilized, Ghayesh, Païdoussis & Amabili [8] proposed an expression for U along the pipe length based on the conservation of the volumetric flow rate inside the pipe

$$U = U[q_i] = \frac{1 + \varepsilon[q_i]}{1 + b\varepsilon[q_i]} U_0, \tag{7}$$

where ε is the axial strain. As ε is a function of the generalized coordinates of the problem, so is U.
In the present work, the authors adopt the conservation of mass for incompressible flows applied in an infinitesimal volume of the tube (Fig. 1b) to obtain an generalized form for U. Consider that its length is $\mathrm{d}s$ and the velocities upstream and downstream are U and $U + \mathrm{d}U = U + \frac{\partial U}{\partial s}\mathrm{d}s$, respectively. Using the definition of $\frac{\partial U}{\partial s}\mathrm{d}s = \frac{\partial U}{\partial s}\frac{\partial s}{\partial x_0}\mathrm{d}x_0 = \frac{\partial U}{\partial x_0}\mathrm{d}x_0$, $\varepsilon = \frac{\mathrm{d}s - \mathrm{d}x_0}{\mathrm{d}x_0}$ and area variation due to Poisson effect through the volume change rate b, it can be proven that

$$U = U[\dot{q}_i; q_i] = U_0 - \int_0^{x_0} \frac{b(1 + \varepsilon[q_i])}{1 + b\varepsilon[q_i]} \frac{\partial \varepsilon[q_i]}{\partial t}\, \mathrm{d}x_0. \tag{8}$$

Notice that the expression above is a function of both the generalized coordinates and velocities of the problem.

Derivation of the equations of motion

The equations of motion in the continuum are formulated up to polynomial cubic-order terms with Equation (6) and discretized with Galerkin's method, originating ODEs. The kinetic energy T_u accounts for the tube and fluid particles, so

$$T_u = \frac{m}{2}\int_0^L \dot{\mathbf{R}} \cdot \dot{\mathbf{R}}\, \mathrm{d}x_0 + \frac{M}{2}\int_0^L \left(\dot{\mathbf{R}} + U\boldsymbol{\tau}\right) \cdot \left(\dot{\mathbf{R}} + U\boldsymbol{\tau}\right)\, \mathrm{d}x_0. \tag{9}$$

Suppose, for the sake of simplicity, that the work of the nonconservative forces is identically null, thus $W = -V$, in which V is the potential energy related to the deformation of the pipe and the gravitational field

$$V = \frac{EA}{2}\int_0^L \varepsilon^2 \, \mathrm{d}x_0 + \frac{EI}{2}\int_0^L (1+\varepsilon)^2\kappa^2 \, \mathrm{d}x_0 - (m+M)g\int_0^L x \, \mathrm{d}x_0. \tag{10}$$

Considering the nonmaterial volume, we can write $\mathbf{v} = \dot{\mathbf{R}} + U\boldsymbol{\tau}$ and $\mathbf{u} = \dot{\mathbf{R}}$, therefore, $\delta\mathbf{p} = \left(\frac{\partial \dot{\mathbf{R}}}{\partial \dot{q}_i} + \boldsymbol{\tau}\frac{\partial U}{\partial \dot{q}_i}\right)\delta q_i$ and $\delta\mathbf{r} = \frac{\partial \dot{\mathbf{R}}}{\partial \dot{q}_i}\,\delta q_i$. The transport terms are nonzero only if $x_0 = L$, so

$$\int_{\partial V_u} \rho\,(\mathbf{v}\cdot\delta\mathbf{p})\,(\mathbf{v}-\mathbf{u})\cdot\mathbf{n}\,\mathrm{d}\partial V_\mu = MU\left[\left(\dot{\mathbf{R}} + U\boldsymbol{\tau}\right)\cdot\left(\frac{\partial \dot{\mathbf{R}}}{\partial \dot{q}_i} + \boldsymbol{\tau}\frac{\partial U}{\partial \dot{q}_i}\right)\right](\boldsymbol{\tau}\cdot\boldsymbol{\tau})\,\delta q_i\Bigg|_{x_0=L}, \tag{11}$$

$$\int_{\partial V_u} \frac{1}{2}\,\rho\,(\mathbf{v}\cdot\mathbf{v})\,(\delta\mathbf{p}-\delta\mathbf{r})\cdot\mathbf{n}\,\mathrm{d}\partial V_u = \frac{M}{2}\frac{\partial U}{\partial \dot{q}_i}\left[\left(\dot{\mathbf{R}} + U\boldsymbol{\tau}\right)\cdot\left(\dot{\mathbf{R}} + U\boldsymbol{\tau}\right)\right](\boldsymbol{\tau}\cdot\boldsymbol{\tau})\,\delta q_i\Bigg|_{x_0=L}. \tag{12}$$

A discussion about the term related to the flux of kinetic energy - Eq. (12) - is also found in Kheiri & Païdoussis [5]. This term relies on the value of $\delta\mathbf{p} - \delta\mathbf{r} = \boldsymbol{\tau}\frac{\partial U}{\partial \dot{q}_i}\,\delta q_i$. If the pipe is ideally inextensible, $U = U_0$ and $\frac{\partial U}{\partial \dot{q}_i} = 0$. When the extensibility condition is utilized as in Ghayesh, Païdoussis & Amabili [8], Equation (7), the velocity U is not a function of the generalized velocities, therefore, it still cancels out. The present research proposes Equation (8) as an expression for U. Under this condition, $\frac{\partial U}{\partial \dot{q}_i} \neq 0$.
The polynomial cubic-order equations of motion in the continuum are derived following Variational Calculus techniques. Considering established dimensionless quantities well defined in the literature listed in Table 1, the PDEs can be rewritten and discretized via Galerkin's method.

Table 1: Dimensionless quantities.

Dimensionless parameter	**Symbol**	**Definition**
Undeformed coordinate	ξ	$\frac{x_0}{L}$
Axial coordinate and displacement	$\hat{x}$, $\hat{u}$	$\frac{x}{L}$, $\frac{u}{L}$
Transversal coordinate and displacement	$\hat{z}$, $\hat{w}$	$\frac{z}{L}$, $\frac{w}{L}$
Time	τ	$\left(\frac{EI}{M+m}\right)^{1/2}\frac{t}{L^2}$
Axial and flexural stiffness ratio	α	$\frac{EA}{EI}L^2$
Internal flow velocity	v	$\left(\frac{M}{EI}\right)^{1/2}U_0L$
Quotient between linear masses	β	$\frac{M}{M+m}$
Gravitational and flexural stiffness ratio	γ	$\frac{M+m}{EI}gL^3$

For the proposed discretization, the ODEs are obtained assuming

$$\hat{u}[\xi;\tau] \cong \sum_{k=1}^{N} \psi_k[\xi]\hat{u}_k[\tau], \tag{13}$$

$$\hat{w}[\xi;\tau] \cong \sum_{k=1}^{N} \psi_k[\xi]\hat{w}_k[\tau], \tag{14}$$

in which ψ_k represents the family of projection functions

$$\psi_k[\xi] = \sqrt{2}\left(\frac{1-\cos\Lambda_k\xi}{\Lambda_k}\right). \tag{15}$$

Each value Λ_k results from the characteristic equation $\cos\Lambda_k = 0$. With these definitions, fixed end boundary conditions $\hat{u}[\xi=0;\tau] = 0$, $\hat{w}[\xi=0;\tau] = 0$ and $\frac{\partial\hat{w}}{\partial\xi}[\xi=0;\tau] = 0$ are satisfied.

Results

Through a process of linearization of the corresponding nonlinear ODEs around the static equilibrium configuration - using v as a control parameter - root loci diagrams can be obtained and the stability of the system evaluated with the determination of the critical velocity v_{crit} by assessing the real part of the eigenvalues, following Lyapunov's indirect method. Numerical integration is utilized to verify the dynamic behavior observed in the Argand's type diagrams.
The nonlinear equations of motion of the inextensible pipe conveying fluid obtained by Semler, Li & Païdoussis [9] are also discretized with Equation (14) and analyzed. These results are compared with the proposed extensible model, which is consistent with the continuity equation, and the extensible system found in Ghayesh, Païdoussis & Amabili [8].
Two distinct scenarios of Table 3 are investigated with $N = 8$. The values are commonly used in the technical literature.

Table 2: Chosen scenarios.

Scenarios	β	α	ν	b	γ
S1	0.2	1000	0.5	0	0
S2	0.2	1000	0.5	0	100

It is important to note that the static equilibrium configuration for the transversal displacement is equal to zero because no external forces are assumed. For the extensible models, the axial static displacement is a function of γ and v, and is depicted with a graphic of $u \times v$ for some cross sections of the pipe; while the static displacement is identically zero in the inextensible case.

Root loci of the linearized models

The root loci diagrams show how each period T_i evolves with v, also displaying a color scale related to the value of the real part of the corresponding eigenvalue λ_i , i.e., characterizing the stability of the system in the neighborhood of the static equilibrium points. The eigenvalues and periods related to the transversal modes are shown in the inextensible diagrams. In the extensible ones, the associated axial quantities are perceived.
Figures 2 and 3 illustrate the stability analysis of **S1**. According to Figures 2, 3b and 3d, a Hopf bifurcation in the second transversal mode occurs at approximately the same value, defining the critical velocity $v_{crit} \approx 5.6$. The curves associated with the transversal modes are similar, with discrepancies for higher values of v in the post-critical interval, related to the order of the terms utilized in the derivation of each set of equations of motion and their linearizations. The axial periods present in the extensible diagrams remain almost constant. The main difference between the extensible models can be found in the axial static equilibrium configuration of Figures 3a and 3c: with the increase of v, a contraction is present in all the cross sections of the proposed extensible model, while in the results from Ghayesh, Païdoussis & Amabili [8], there is no static displacement.

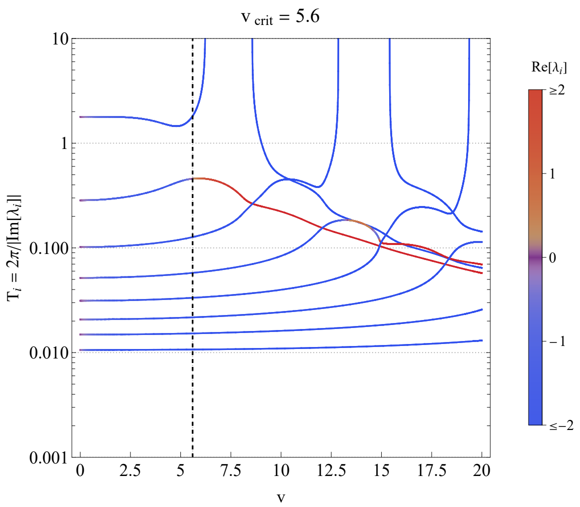
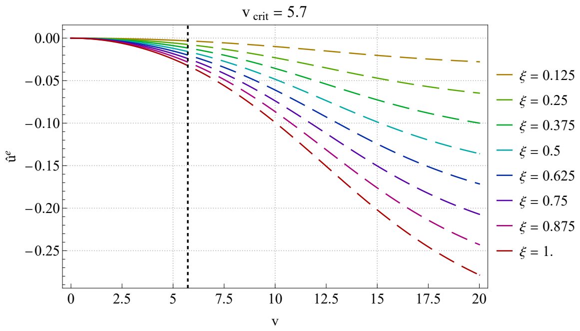


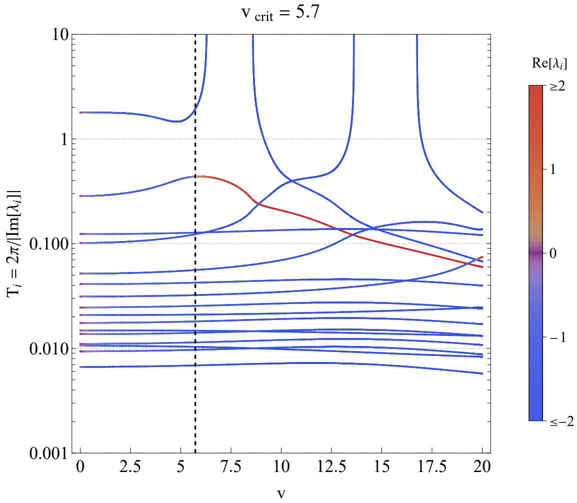
Figure 2: Root locus of the inextensible model for Scenario **S1**.

The stability analysis of Scenario **S2** is presented in Figures 4 and 5. The gravitational effects associated to $\gamma \neq 0$ are well documented in the literature, thus, a higher critical velocity is expected and noticed. In all the models, a Hopf bifurcation happens in the third transversal mode. For the inextensible model and the extensible model formulated in [8], Figures 4 and 5d, $v_{crit} \approx 10.5$, whereas in the proposed extensible model, Figure 5b, $v_{crit} \approx 11.3$, a significantly different value. The periods related to the axial modes are still relatively constant with v.

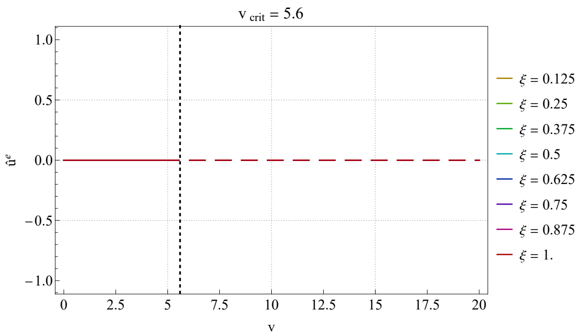
The behavior of the axial static configurations of Figures 5a and 5c are remarkably distinct: as $\gamma > 0$, there is a pipe elongation present in both of the graphs at $v = 0$, but a contraction in all the cross sections exists for the proposed extensible model and the elongation becomes higher in the case of the model based on [8].



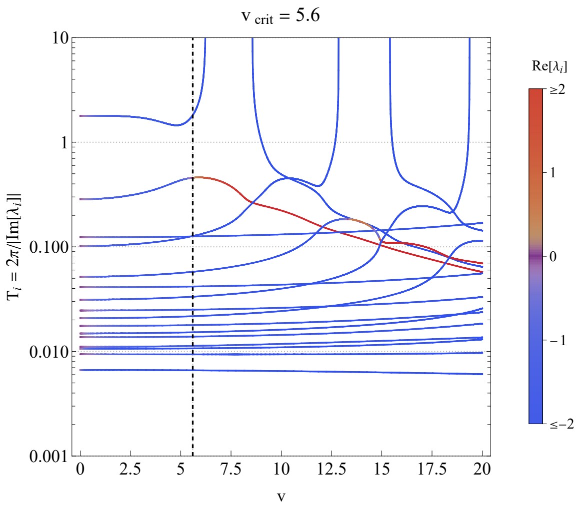
(a) Axial static equilibrium configuration of the proposed extensible model for some cross sections of the pipe.



(b) Root locus of the proposed extensible model.



(c) Axial static equilibrium configuration of the model derived in [8] for some cross sections of the pipe.



(d) Root locus of the model derived in [8].

Figure 3: Stability analysis of the extensible models for Scenario **S1**.

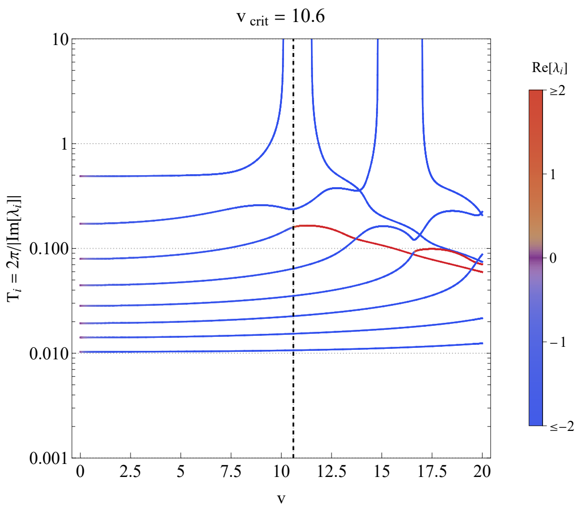
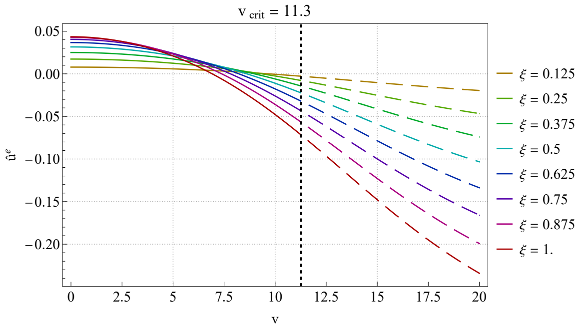
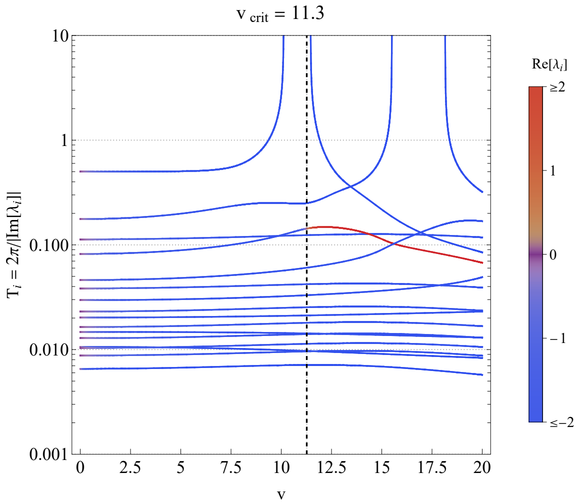


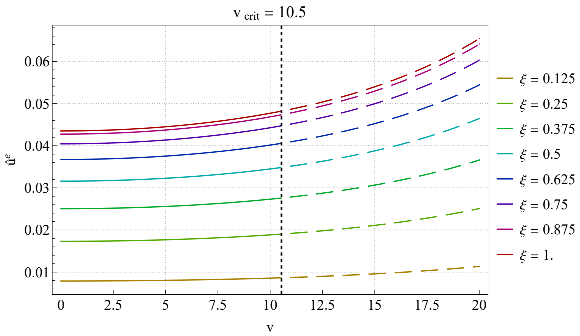
Figure 4: Root locus of the inextensible model for Scenario **S2**.



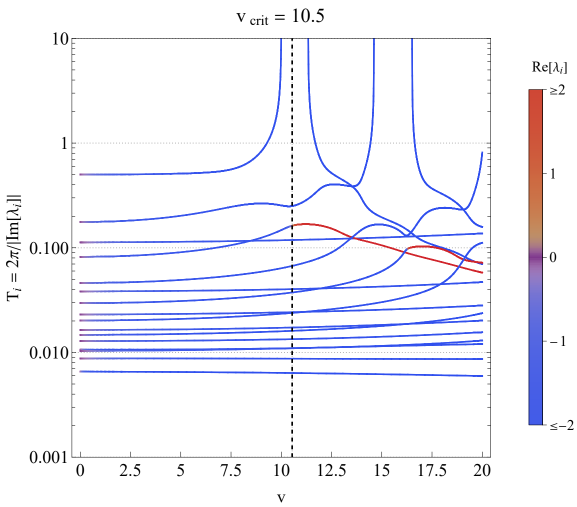
(a) Axial static equilibrium configuration of the proposed extensible model for some cross sections of the pipe.



(b) Root locus of the proposed extensible model.



(c) Axial static equilibrium configuration of the model derived in [8] for some cross sections of the pipe.

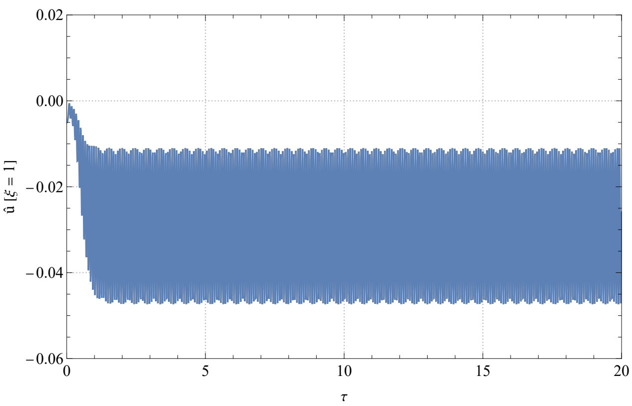


(d) Root locus of the model derived in [8].

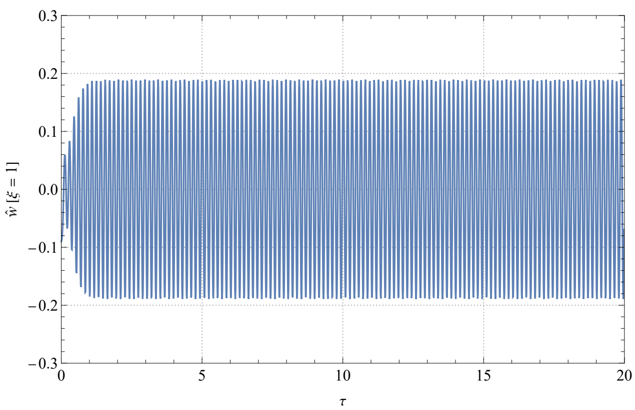
Figure 5: Stability analysis of the extensible models for Scenario **S2**.

Numerical integration

Numerical integration of the nonlinear equations of motion of Scenario **S2** at $v = 11$ is done to verify the different behaviors foreseen from the root loci: for the ideally inextensible model and the extensible model of Ghayesh, Païdoussis & Amabili [8], there is a dynamic instability denoted by limit cycles; for the extensible model proposed in this work, the root loci indicate stability in the vicinity of the static equilibrium points. The free end ($\xi = 1$) axial and transversal displacements are shown in the Figures 6 and 7.

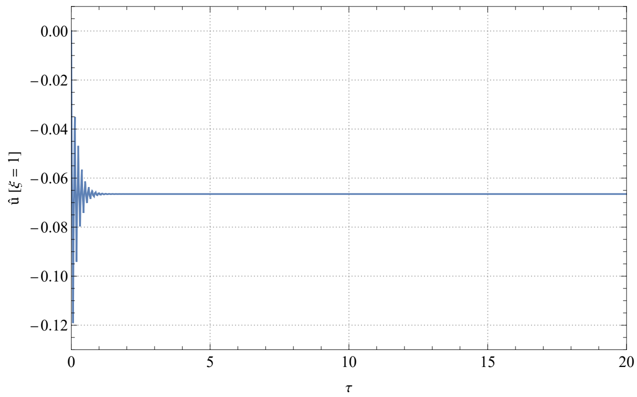


(a) $\hat{u}[\xi = 1; \tau] \times \tau$.

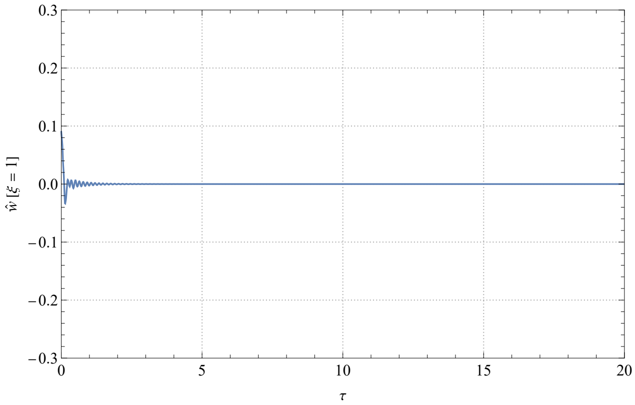


(b) $\hat{w}[\xi = 1; \tau] \times \tau$.

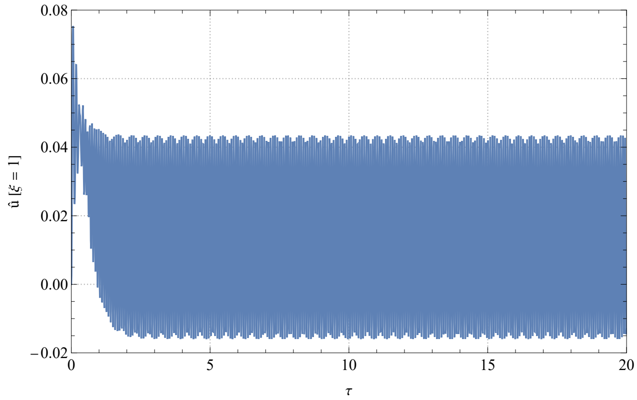
Figure 6: Numerical integration of Scenario **S2** at $v = 11$ of the inextensible model shows dynamic oscillatory instability.



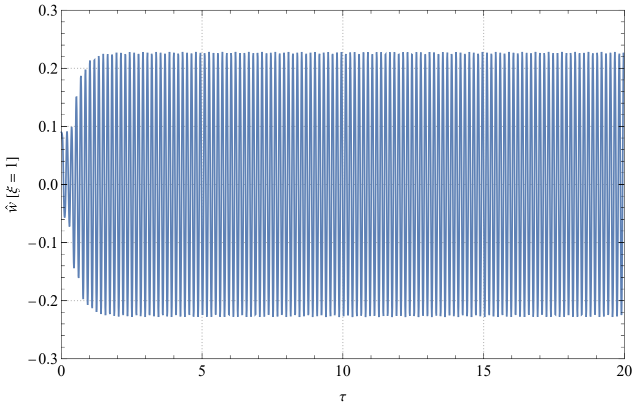
(a) $\hat{u}[\xi = 1; \tau] \times \tau$ of the proposed extensible model.



(b) $\hat{w}[\xi = 1; \tau] \times \tau$ of the proposed extensible model.



(c) $\hat{u}[\xi = 1; \tau] \times \tau$ of the model derived in [8].



(d) $\hat{w}[\xi = 1; \tau] \times \tau$ of the model derived in [8].

Figure 7: Numerical integration of Scenario **S2** at $v = 11$ of the proposed extensible model shows stability around the static equilibrium, while Ghayesh, Païdoussis & Amabili [8] extensible model displays a post-critical oscillatory regime.

The results obtained in time domain are consistent with those predicted by the linear analysis. Post-critical oscillatory regimes are noticed in the graphics of the inextensible and Ghayesh, Païdoussis & Amabili [8] models, whereas the proposed extensible model - which satisfies the conditions of the conservation of mass - shows stability.

Concluding Remarks

In this paper, a 2D reduced-order model for the cantilevered extensible pipe conveying fluid problem is proposed, in which the necessary conditions for conservation of mass (continuity equation) are satisfied. This model is obtained using the Extended Hamilton's Principle for nonmaterial volumes as formulated in Casetta & Pesce [4], with a cubic-order polynomial truncation and approximation followed by a discretization procedure based on Galerkin's method.

Although the formulation of the extensible model has similarities with nonlinear models of large displacements and small strains much discussed in the technical literature, the main difference is the consistent consideration of the conservation of mass in the internal flow. With such a hypothesis, a closed-form expression along the pipe length for the internal velocity U can be derived, which is a function of both the generalized coordinates and velocities of the problem (Eq. (8)). Previously, an extensible model discussed in Ghayesh, Païdoussis & Amabili [8] utilized Equation (7), in which the internal velocity U only depends on the generalized coordinates.

These distinct expressions for U have implications in the term related to flux of kinetic energy present in the Extended Hamilton's Principle, the fourth term of Equation (6). Under the assumption of an ideally inextensible pipe or Ghayesh, Païdoussis & Amabili [8] extensible model, this term is identically zero and the Hamilton's Principle for nonmaterial volumes is reduced to a form first obtained in McIver [2], Eq. (4). With the consideration of the proposed extensible model, the term associated with the transport of kinetic energy is nonzero and scenarios are investigated to study its influence on the dynamic response of the system.

Two tools are utilized for the comparison of the inextensible equations of motion of Semler, Li & Païdoussis [9], the extensible model described in this work and the extensible one obtained in Ghayesh, Païdoussis & Amabili [8]: (i) root loci diagrams calculated in the neighborhood of the static equilibrium configuration with v as control parameter and the characterization of the stability condition through Lyapunov's indirect method; (ii) numerical integration of the discretized nonlinear equations of motion to verify the dynamic behavior predicted by the root loci diagrams.

The analysis indicate a similar response for Scenario **S1** for all the models, with a Hopf bifurcation associated with the second transversal mode determining a critical velocity of $v_{crit} \approx 5.6$. The axial static equilibrium configuration of the

extensible models are distinct: for the proposed model, there is a contraction with the increasing of v, whereas for the extensible one derived in [8], the static displacement is zero.
For Scenario **S2**, a Hopf bifurcation occurs at the third transversal mode, but for the consistent extensible model $v_{crit} \approx 11.3$, and for the others, $v_{crit} \approx 10.5$. The root loci diagrams show a deviation of the present critical velocity with the numerical integration, confirming the expected response at $v = 11$ for the models. The axial static equilibrium remains different for the two extensible pipes.
Thus, extensibility effects may have substantial practical implications in the dynamics of this kind of system. The results obtained with the proposed extensible model reveal a higher critical velocity in a given scenario and different axial static configurations when compared to the cited previous models.

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