# Resonance Steady State and Transient in the Non-Ideal System having the Pendulum Absorber

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*Summary*. Resonance behavior of the system with a limited power-supply (or non-ideal system) having the pendulum as absorber is considered. The multiple scales method is used to describe the system dynamics near the resonance. It is shown that the essential reduction of the resonance vibration amplitudes can be obtained by choose of the system parameters. Transient in the non-ideal system under consideration is effectively constructed using the rational Padé approximation. Tending of the transient to the resonance steady state is shown. It is shown that the amplitudes of resonant oscillations of the elastic subsystem can be essentially reduced by choosing some system parameters.

## Introduction. The basic model.

The systems with limited power supply are characterized by interaction of the source of energy and elastic sub-system which is under action of the source. Such systems are named also as non-ideal systems (NIS). For the NIS the external excitation depends on the excited elastic sub-system dynamics. The most interesting effect appearing in non-ideal systems is the Sommerfeld effect [1], when in the elastic sub-system the large amplitude resonance regime is appeared, and the big part of the vibration energy passes from the energy source to the resonance behavior. Resonance dynamics of the systems with limited power supply is first analytically described by V.Kononenko [2]. Then investigations on the subject were continued by Kononenko [3] and other authors [4-7]. Reviews of numerous studies of the NIS dynamics can be found in [8-10]. We can note that different types of the NIS behaviour were considered, including forced and parametric oscillations, self-oscillations, transient, chaotic oscillations, interaction of the NIS with energy sources of different physical characteristics, and so on.

It is known that nonlinear vibration absorbers can significantly reduce the amplitudes of resonant elastic vibrations. We consider here the resonant behaviour of the non-ideal system with three DOF (Fig.1), having the pendulum-type absorber, by the multiple scales method. Both the resonance steady state and the transient are constructed. The transient is effectively presented using the rational Padé approximants [11] containing exponents. It is shown that amplitudes of the resonant oscillations of the elastic subsystem can be reduced by changing some system parameters.

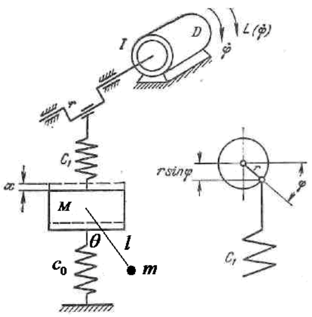


Figure. 1. The model under consideration

## Resonance steady state solution. Influence of the system parameters to resonance dynamics of the system

Equations of motion of the system under consideration with respect to variables $x$, $\varphi$ and $\theta$ are the following:

$$\begin{cases} (M+m)\ddot{x}+(c_0+c_1)x=c_1 r\sin\varphi - ml\left(\ddot{\theta}\cos\theta-\dot{\theta}^2\sin\theta\right); \\ I\ddot{\varphi}=a-b\dot{\varphi}+c_1 r\left(x-r\sin\varphi\right)\cos\varphi; \\ ml\left(l\ddot{\theta}+g\sin\theta+\ddot{x}\cos\theta\right)=0. \end{cases} \quad (1)$$

Here $I$ is the moment of inertia of rotating masses; $(c_0+c_1)$ is the rigidity of the elastic subsystem having the mass $M$; the combination $L=a-b\dot{\varphi}$ describes both the driving moment of the energy source and the moment of the forces of resistance to the rotation that is the so-called characteristics of the engine. From equations (1) it is seen that the moment $c_1 rx\cos\varphi$ is the part of the motor excitation that depends on the oscillations of the elastic subsystem.

### Construction of a stationary resonant solution.

In the first place, we use some transformations. Namely, the functions $\cos\theta$ and $\sin\theta$ are expanded in the McLaren series, and terms remain to the third degree. Then a small parameter $\varepsilon$, introduced into the equations of motion, characterizes the small mass of the absorber with respect to the mass of the elastic part of the system, $m \to \varepsilon m$, and the smallness of vibration components in variability in time of the angle $\varphi$ velocity with respect to its main constant component. Terms $\varepsilon h\dot{x}$ and $\varepsilon h\dot{\theta}$ describe the small dissipation. Considering a region of the resonance between frequencies of the motor rotation and the elastic sub-system vibrations, we introduce the small frequency detuning as $\varepsilon\Delta = \omega_x^2 - \dot{\varphi}^2$, where $c_0 + c_1 = M\omega_x^2$. We also assume that in the resonance region the external excitation of the elastic subsystem is small. A relatively not large nonlinear part of the elastic subsystem response is represented by the term $\varepsilon\tau x^3$, which is included in the first equation of the system (1). As a result, we consider the following equations of motion instead of the equations (1):

$$\begin{cases} (M+\varepsilon m)\ddot{x} + \omega_x^2 x + \varepsilon h\dot{x} + \varepsilon\tau x^3 = \varepsilon c_1 r\sin\varphi - \varepsilon ml\left(\ddot{\theta}\left(1-\frac{1}{2}\theta^2\right) - \left(\theta - \frac{\theta^3}{6}\right)\dot{\theta}^2\right) \\ I\ddot{\varphi} = \varepsilon\left(a - b\dot{\varphi} + c_1 r\left(x - r\sin\varphi\right)\cos\varphi\right) \\ \varepsilon ml\left(l\ddot{\theta} + g\left(\theta - \frac{\theta^3}{6}\right) + \ddot{x}\left(1-\frac{1}{2}\theta^2\right)\right) + \varepsilon h\dot{\theta} = 0 \end{cases} \tag{2}$$

The multiple scales method [12] is used to describe the behaviour of the system in the field of resonance. According to this method, we write the following representations of solutions:

$$x(t,\varepsilon) = x\left(t, \varepsilon t, \varepsilon^2 t, ...; \varepsilon\right); \quad \varphi(t,\varepsilon) = \varphi\left(t, \varepsilon t, \varepsilon^2 t, ...; \varepsilon\right); \quad \theta(t,\varepsilon) = \theta\left(t, \varepsilon t, \varepsilon^2 t, ...; \varepsilon\right) \tag{3}$$

In addition, the following transformations are used:

$$\begin{aligned} &\frac{d}{dt} = \frac{\partial}{\partial T_0} + \varepsilon\frac{\partial}{\partial T_1} + \varepsilon^2\frac{\partial}{\partial T_2} + ... \\ &\frac{d^2}{dt^2} = \frac{\partial^2}{\partial T_0^2} + 2\varepsilon\frac{\partial}{\partial T_0}\frac{\partial}{\partial T_1} + \varepsilon^2\left(\frac{\partial^2}{\partial T_1^2} + 2\frac{\partial}{\partial T_0}\frac{\partial}{\partial T_2}\right) + ... \end{aligned} \tag{4}$$

Here $T_0 = \omega t,\ T_1 = \varepsilon\omega t$ and $\omega = \omega_x$. The representations (3) are decomposed in the form of the power series by the small parameter. Substituting the power series into the system (2), we distinguish the terms of zero and first degrees by the small parameter. As a result, the following system of differential equations can be obtained:

$$\varepsilon^0 : \frac{\partial^2 x_0}{\partial T_0^2} + \Omega^2 x_0 = 0, \tag{5}$$

$$\frac{\partial^2 \varphi_0}{\partial T_0^2} = 0, \tag{6}$$

$$\begin{aligned} &\varepsilon^1 : 2M\frac{\partial^2 x_0}{\partial T_0 \partial T_1} + M\frac{\partial^2 x_1}{\partial T_0^2} + m\frac{\partial^2 x_0}{\partial T_0^2} + \Delta M x_0 + M\Omega^2 x_1 + h\frac{\partial x_0}{\partial T_0} + r\tau x_0^3 = c_1 r\sin\left(\Omega T_0\right) - \\ &-ml\frac{\partial^2\theta_0}{\partial T_0^2} + \frac{1}{2}ml\frac{\partial^2\theta_0}{\partial T_0^2}\cdot\theta_0^2 + ml\theta_0\left(\frac{\partial\theta_0}{\partial T_0}\right)^2 - ml\frac{\theta_0^3}{6}\cdot\left(\frac{\partial\theta_0}{\partial T_0}\right)^2 , \end{aligned} \tag{7}$$

$$2\frac{\partial^2\varphi_0}{\partial T_0 \partial T_1} + \frac{\partial^2\varphi_1}{\partial T_0^2} = K - N\Omega\frac{\partial\varphi_0}{\partial T_0} + qx_0\cos\varphi_0 - \frac{qr}{2}\sin 2\varphi_0, \tag{8}$$

$$l\frac{\partial^2\theta_0}{\partial T_0^2} + g\theta_0 - g\frac{\theta_0^3}{6} + \frac{\partial^2 x_0}{\partial T_0^2} - \frac{\theta_0^2}{2}\frac{\partial^2 x_0}{\partial T_0^2} + \tilde{h}\frac{\partial\theta_0}{\partial T_0} = 0 \tag{9}$$

Here $\tilde{h} = h/(ml)$, $\Omega$ is the frequency of the motor rotation, depending on the time scale $T_1$. The solutions of equations of the zero approximation by the small parameter (5) and (6) are presented as follows:

$$\begin{cases} x_0 = A\cos\left(\Omega T_0\right) + B\sin\left(\Omega T_0\right) \\ \varphi_0 = \Omega T_0 \end{cases} \tag{10}$$

We assume that in the resonance between the engine speeds and the oscillations of the elastic subsystem, the amplitudes of the pendulum oscillations are small. Therefore, we assume that in equation (9) all members with a degree greater than one have the order of the small parameter $\varepsilon$. As a result, from equation (9) we have the following:

$$\theta_0 = C\,cos\,\Omega T_0 + D\,sin\,\Omega T_0, \quad \text{where} \quad C = \frac{A\Omega^2}{g - l\Omega^2}, \quad D = \frac{B\Omega^2}{g - l\Omega^2}. \tag{11}$$

The solution of the zero approximation (10) is substituted to the equation (7). To avoid the appearance of secular terms the following modulation equations are obtained:

$$2\frac{\partial A}{\partial T_1}\Omega + \mu B\Omega^2 - \Delta B + \frac{hA\Omega}{M} - \frac{3\tau A^2 B}{4M} - \frac{3\tau B^3}{4M} + \sigma + \mu l D\Omega^2 = 0, \tag{12}$$

$$-2\frac{\partial B}{\partial T_1}\Omega + \mu A\Omega^2 - \Delta A - \frac{hB\Omega}{M} - \frac{3\tau AB^2}{4M} - \frac{3\tau A^3}{4M} + \mu l C\Omega^2 = 0, \tag{13}$$

$$\left(\mu = \frac{m}{M}, \sigma = \frac{c_1 r}{M}\right)$$

To avoid the appearance of secular terms in the solution of the equation (8) we use the following relation:

$$-2\frac{\partial\Omega}{\partial T_1} + K - N\Omega^2 + \frac{qA}{2} = 0, \quad \text{where} \quad K = \frac{a}{I\omega_x^2}, \quad N = \frac{b}{I\omega_x}, \quad q = \frac{c_1 r}{I\omega_x^2}. \tag{14}$$

Together, all three equations (12-14) give variables $A$, $B$ and $\Omega$, which correspond to the resonant state. Considering the steady state, we assume that the values of $A$, $B$ and $\Omega$ are constant. In this case, equations (12-14) are transformed into a system of nonlinear algebraic equations for these values, which is solved by the Newton's numerical method. Thus, constants can be obtained for a stationary solution $\tilde{A}_0, \tilde{B}_0, \tilde{\Omega}_0$. In particular, from equation (14) we have that

$$\tilde{\Omega}_0 = \pm((2a + c_1 r\tilde{A}_o)/2b)^{1/2} \tag{15}$$

Note that in the resonance region, the frequencies $\Omega$ and $\omega_x$ differ by an order of magnitude of the small parameter $\varepsilon$. Thus, if in the coefficients *K*, *N* the variable frequency $\Omega$ changes by $\omega_x$, then we can find the following solution of equation (14):

$$\Omega = \tilde{\Omega}_0 + \rho\left(\eta + 1\right), \quad \text{where} \quad \rho = C_2 = \frac{\tilde{\Omega}_0}{10}, \quad \eta = e^{-\frac{N}{2}T_1} - 1. \tag{16}$$

The last relation shows the approach of the motor frequency to the stationary value of $\tilde{\Omega}_0$ with increasing time.

## Construction of the transient using the Padé approximants.

The expression (16) is substituted to equations (12) and (13), preserving the terms of zero and first degree by the variable $\boldsymbol{\eta}$. To solve the obtained differential equations, the following representations of functions $A$ and $B$ in the form of the following power series:

$$A = A_o + A_1\eta + A_2\eta^2 + \cdots \quad , \quad B = B_o + B_1\eta + B_2\eta^2 + \cdots \tag{17}$$

Here the magnitudes $A_0$ and $B_0$ selected to match the corresponding values for stationary mode, namely, the values $\tilde{A}_0, \tilde{B}_0$. Next, we need to select equations that contain members of the order $\eta^0, \eta, \ldots$ The zero approximation solutions with respect to $\eta$, which is not presented here, permits to obtain $A_1$ and $B_1$, namely

$$A_1 = \frac{\mu B_0 \Omega_0}{2\beta} + \frac{\mu B_0 \rho}{2\beta} - \frac{\omega_x^2 B_0}{2\beta(\Omega_0+\rho)} + \frac{B_0 \Omega_0}{2\beta} + \frac{B_0 \rho}{2\beta} + \frac{hA_0}{2\beta M} - \\ - \frac{3\tau A_0^2 B_0}{8\beta M(\Omega_0+\rho)} - \frac{3\tau B_0^3}{8\beta M(\Omega_0+\rho)} + \frac{\sigma}{2\beta(\Omega_0+\rho)} + \frac{\mu l B_0 (\Omega_0+\rho)^3}{2\beta G}; \tag{18}$$

$$B_1 = -\frac{\mu A_0 \Omega_0}{2\beta} - \frac{\mu A_0 \rho}{2\beta} + \frac{\omega_x^2 A_0}{2\beta(\Omega_0+\rho)} - \frac{A_0 \Omega_0}{2\beta} - \frac{A_0 \rho}{2\beta} + \frac{hB_0}{2\beta M} + \\ + \frac{3\tau B_0^2 A_0}{8\beta M(\Omega_0+\rho)} + \frac{3\tau A_0^3}{8\beta M(\Omega_0+\rho)} - \frac{\mu l A_0 (\Omega_0+\rho)^3}{2\beta G}. \tag{19}$$

Here $\beta = \frac{N}{2}, G = g - l(\Omega_0+\rho)^2$ . From the equations of the first approximation, i.e. equations containing members of the order $\eta$, the constants $A_2$ and $B_2$ can be found, which are not presented here. Then we introduce the following expansions of functions $A$ and $B$ in power series:

$$A = A_{0in} + A_1\eta + A_2\eta^2 + \ldots \quad B = B_{0in} + B_1\eta + B_2\eta^2 + \ldots \tag{20}$$

Here $(A_{0in}, B_{0in})$ these are here arbitrary values of the amplitudes of oscillations, which are determined by the initial conditions. For further research, we also introduce the following parameter:

$$\psi = \frac{e^{-\beta T_1} - 1}{e^{-\beta T_1}} = \frac{\eta}{\eta+1}, \quad \text{then} \quad \eta = \frac{\psi}{1-\psi} \tag{21}$$

Substituting the relationship (21) to the power series (20) and decomposing these expressions into McLaren series by the parameter $\psi$ , corresponding to the case $T_1 \to 0$ , one has the following:

$$A = A_{0in} + A_1\psi + (A_1 + A_2)\psi + \ldots \\ B = B_{0in} + B_1\psi + (B_1 + B_2)\psi + \ldots \tag{22}$$

We introduce the Padé approximants for values $\psi$, varying it from zero to infinity, corresponding to change of the time scale $T_1$ also from zero to infinity, as:

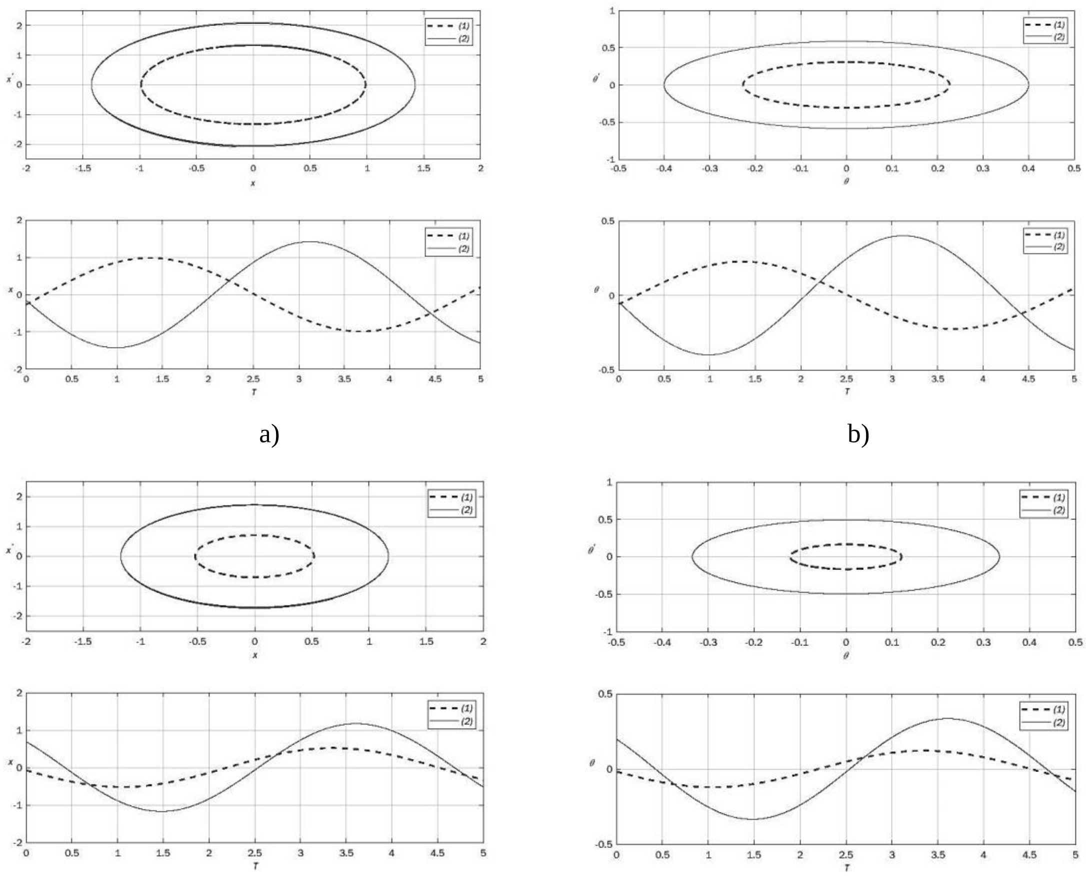
$$A_p = \frac{\alpha_0 + \alpha_1\psi + \alpha_2\psi^2}{1 + \beta_1\psi + \beta_2\psi^2}; \\ B_p = \frac{\tilde{\alpha}_0 + \tilde{\alpha}_1\psi + \tilde{\alpha}_2\psi^2}{1 + \tilde{\beta}_1\psi + \tilde{\beta}_2\psi^2}. \tag{23}$$

Then we compare expressions (23) with series (22). In addition, to describe the approximation of the transition process to the stationary regime, we consider the boundary of expressions (23) when $\psi \to \infty$ (that is, when $T_1 \to \infty$) and equate this limit to the values of the amplitudes $\widetilde{A_0}$ and $\widetilde{B_0}$, previously obtained for stationary mode, i.e. $\frac{\alpha_2}{\beta_2} = \tilde{A}_0, \quad \frac{\tilde{\alpha}_2}{\tilde{\beta}_2} = \tilde{B}_0$. All this makes it possible to obtain coefficients of the Padé approximants (23) from a system of linear algebraic equations.

## Comparative characteristics of the transition and stationary modes. Resonant behaviour of the system when changing system parameters.

Here we consider a comparison of the stationary solution and the transient of the system at small and time values. Then we consider also the influence of the system parameters on the amplitude of elastic oscillations in the resonant region.

This applies to the change of the parameters of the pendulum mass *m* and the parameter of nonlinearity in the elastic force $\tau$. The corresponding numerical simulation was performed for the basic system (2) using the 4th order Runge-Kutta procedure. Change of the driving moment coefficient $a$ and the length of the pendulum $l$ leads to a slight decrease in the amplitude of elastic oscillations, thus graphical representations corresponding to changes in these parameters are not given. From numerical simulations it can be concluded that the amplitudes of resonant elastic oscillations can be significantly reduced with the parameters *m* and $\tau$. We will consider the solutions at different time intervals, as at small values of time, $t \in (0;5)$, and for significant values of time, $t \in (220;225)$. Note that in the pictures a),c) the variable *x(t)*, and in the pictures b),d) the variable *θ(t)* are presented. In all Figs. the following fixed parameters are given: $= 0{,}37261;\ \ l = 1.$ In Fig. 2 the comparison of the stationary solution and the transient at $t \in (0;5)$ is presented for $\tau = 0{,}05.$ In Fig. 2. a,b the parameter $m = 0.07$ and in Fig. 2. c,d one has $m = 0.11$. Fig. 3 shows the comparison of the stationary solution and the transient at $t \in (220;225)$ for the same fixed parameters $a,\ l,\ \tau$ as in Fig. 2. In Fig. 3a,b the parameter $m = 0.07$, and in Fig. 3c,d one has $m = 0.11$ . Fig. 4 presents the comparison of the stationary solution and the transient at $t \in (0;5)$, for *m* = 0.05. In Fig. 4a,b the parameter $\tau = 0{,}01,$ and on Fig. 4c,d one has $\tau = 0{,}05.$ In Fig. 5 the comparison is shown at $t \in (220;225)$ for the same fixes parameters $a,\ l,\ \tau$ as in Fig. 4. In Fig. 5a,b the parameter $\tau = 0{,}01$, and in Fig. 5c,d one has $\tau = 0{,}05$.

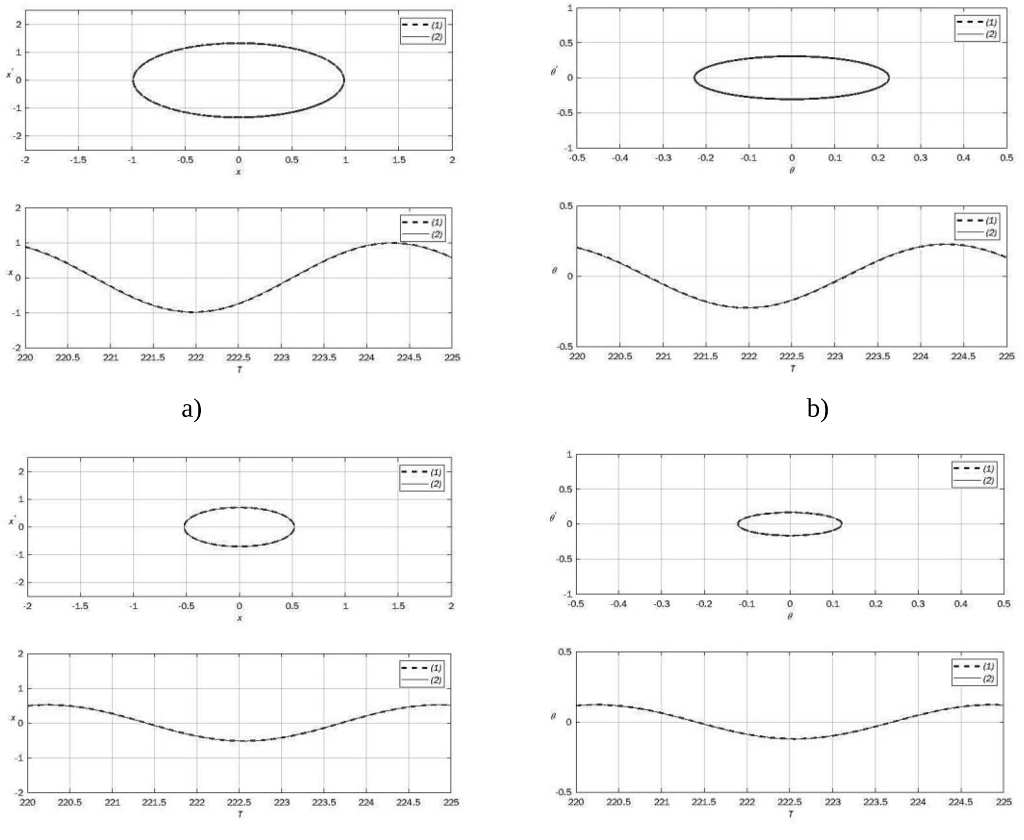


a) b) c) d)

Figure 2. Comparison of the stationary solution (1) and the transient (2) at $t \in (0;5)$: a) variable *x* for $m = 0.07$; b) variable $\theta$ for $m = 0.07$; c) variable *x* for $m = 0.11$; d) variable $\theta$ for $m = 0.11$.

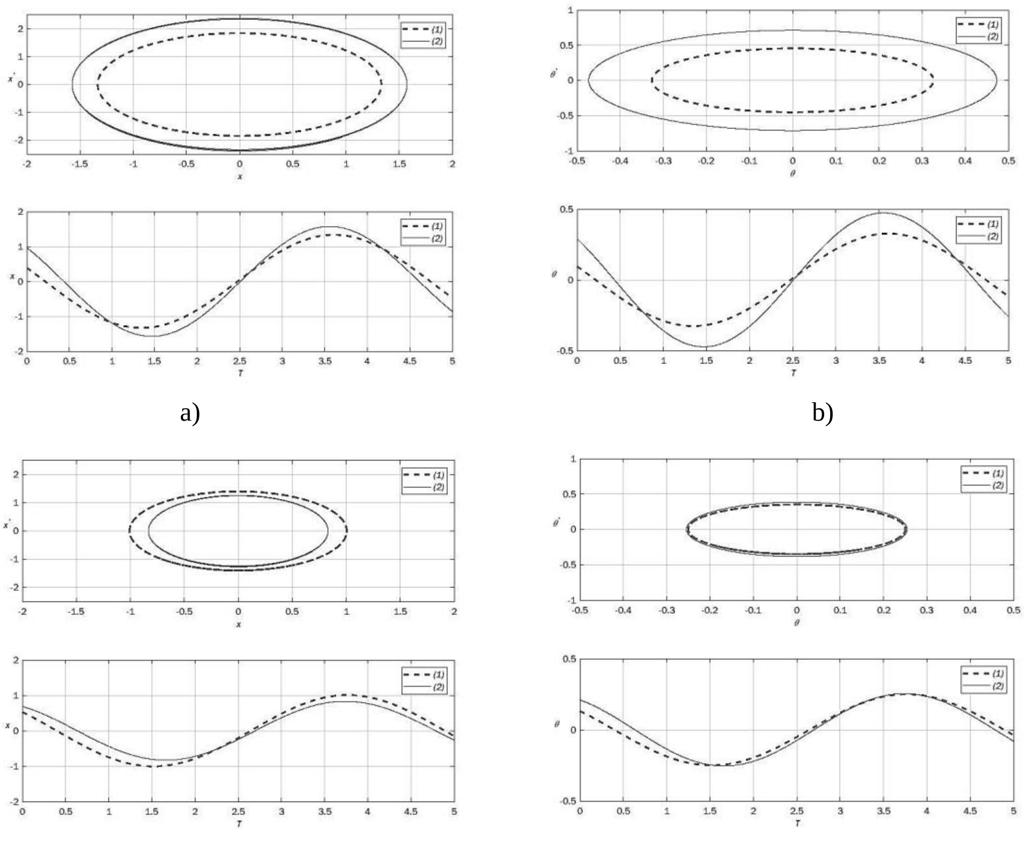
## Conclusions

Analyzing the obtained results, we can obtain the following conclusions. First, it is fashionable to obtain the resonance steady state effectively by the multiple scales method. Secondly, we can see a good coincidence of the transient represented by the Padé approximants to the stationary regime with increasing time values. Thus, the proposed Padé approximants having exponents are very effective for the transient presentation. Finally, the numerical simulation demonstrates a significant decrease in the amplitudes of elastic oscillations with increasing the pendulum mass and the nonlinearity in the elastic force.




a) b) c) d)

Figure 3. Comparison of stationary solution (1) and transient (2) at $t \in (220; 225)$**:** a) variable *x* for $m = 0.07$; b) variable $\theta$ for $m = 0.07$; c) variable *x* for $m = 0.11$; d) variable $\theta$ for $m = 0.11$**.**




a) b) c) d)

Figure 4. Comparison of the stationary solution (1) and the transient (2) at $t \in (0; 5)$: a) variable *x* for $\tau = 0{,}01$; b) variable $\theta$ for $\tau = 0{,}01$; c) variable *x* for $\tau = 0{,}05$; d) variable $\theta$ for $\tau = 0{,}05$.

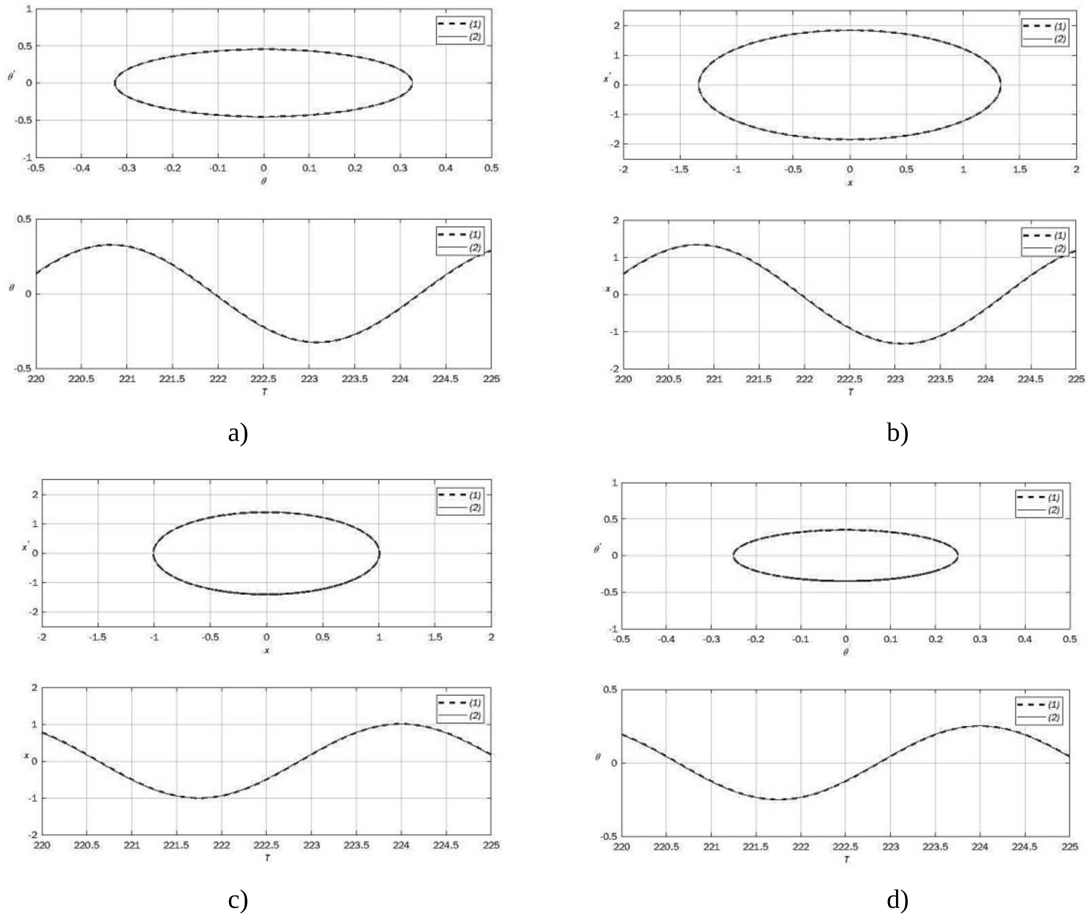



Figure 5. Comparison of the stationary solution (1) and the transient (2) at $t \in (220; 225)$: a) variable *x* for $\tau = 0{,}01$; b) variable $\theta$ for $\tau = 0{,}01$; c) variable *x* for $\tau = 0{,}05$; d) variable $\theta$ for $\tau = 0{,}05$.

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