The hidden bridge between continuous and discontinuous worlds and why period 2 may imply chaos

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Summary. We report a recently developed approach for the investigation of discontinuous maps. Using so-called hidden orbits, we demonstrate that several aspects of the dynamics well-known for continuous maps can also be transferred to discontinuous maps. Moreover, hidden orbits help us to understand the dynamics of maps with steep branches, which are known to be hard to investigate otherwise.

Motivation

Discontinuous maps appear naturally in many areas of nonlinear dynamics. In some situations, they act as approximate models of systems where the rules governing the dynamic behaviour undergo a fast (but continuous) change at some borders in the state space. In other situations, the change of the rules at the boundaries is in fact discontinuous.
It is well known that many properties of discontinuous maps differ quite significantly from the corresponding properties of continuous ones. For example, it is easy to show that the famous rule "period three implies chaos" applies to continuous maps and does not apply to their discontinuous counterparts where a 3-cycle may exist alone. Similarly, bifurcation diagrams one can observe in continuous maps are well organized: for instance, the branches corresponding to stable solutions appearing at (smooth or non-smooth) fold and flip bifurcations are typically connected via branches corresponding to unstable solutions which appear at the same bifurcations and determine the basin boundaries of coexisting attractors. In discontinuous maps, such unstable branches may be missing, the stable solutions may appear at border collision bifurcations "as if from nowhere", and the basins may be separated not only by unstable orbits but also by the discontinuities and their preimages. Chaotic attractors in continuous maps are always cyclic, while in discontinuous maps they may be acyclic as well. Because of these – and many other – differences, it is indeed hard to believe that the two worlds can be unified, i.e., that it is possible to develop an approach dealing with discontinuous maps in such a way that several properties of continuous maps are restored as far as possible.

Approach

A novel approach for the investigation of discontinuous maps has been recently suggested in [1]. The key idea of the approach is to extend the definition of a discontinuous map in such a way that at the discontinuities, the function is considered to be *set-valued* (in particular, for 1D maps, *interval-valued*). It is worth emphasizing that following this approach, the orbits of the map remain single-valued. An orbit visiting a discontinuity is mapped to a point belonging to the corresponding set; if an orbit visits the discontinuity again, it may be mapped to the same or to a different point. Accordingly, in addition to all orbits existing in the original discontinuous map, every time an orbit of such an extended map (referred to as a map with vertical branches) visits a discontinuity, an infinite number of forward orbits (so-called *hidden orbits*) is created. By construction, a *hidden orbit* is an orbit including points inside the discontinuities, and if a hidden orbit $\{x_n \mid n = 1, 2, \dots\}$ satisfies $x_{n+p} = x_n$ for all n, the orbit forms a *hidden cycle* of period p. Clearly, each hidden cycle is repelling and can easily be computed, as its points are given by preimages of the corresponding discontinuity.

Results

There are several different application areas for the proposed approach.

- When dealing with all kinds of discontinuous maps, the corresponding maps with vertical branches simplify the bifurcation analysis by adding to the bifurcation diagrams the "missing" unstable branches given by hidden cycles. In this way, by the bifurcation structures in discontinuous maps can be described in terms well-known for continuous maps: for example, the border collision bifurcations at which a cycle appears "as if from nowhere" turn into the usual border collision flip and fold bifurcations [2]. Hidden orbits unify also the treatment of basin boundaries: if in a discontinuous map these boundaries are given by a discontinuity and its preimages, in a corresponding map with vertical branches there is a (repelling) hidden cycle at the basin boundary, similarly to continuous maps.

- A discontinuous map may act as a model of a system with a very fast but continuous switching process. In such cases, a more detailed modeling leads to maps with steep branches which are quite hard to deal with (from the numerical point of view, but also because the laws governing the fast switching process are not always known). Here, a map with vertical branches provides an approximation for dynamics involving steep branches. Clearly, a cycle including points on a steep branch is strongly repelling and hard to find numerically. By contrast, to calculate a corresponding hidden cycle is a simple task, as it is given by a sequence of preimages of the discontinuity.

- By definition, a map with vertical branches is discontinuous but connected. Several fundamental theorems have been proven for continuous maps and do not apply to discontinuous ones. However, one may ask whether the requirement for continuity of the function may be relaxed and whether the connectedness would be sufficient as well. This is the case for the Sharkovsky theorem (which implies, in particular, the well-known rule "period three implies chaos"): this theorem can be proven not only for continuous but also for maps with vertical branches [3]. In this way, hidden cycles restore the Sharkovsky ordering, providing all cycles which are missing in the usual discontinuous map (without vertical branches). On the other hand, if one can prove that a hidden cycle of a certain period does not exist in a map with vertical branches, then a non-hidden cycle of this period must exist in the corresponding discontinuous map.

- A striking property of hidden orbits is that the existence of two distinct hidden cycles implies that a countable number of other hidden cycles and an uncountable number of hidden aperiodic orbits exist as well. Although all these orbits may be located at a final number of points in the state space (the points of discontinuities and their preimages), their union can be seen as a hidden chaotic repeller. It is worth noting that under perturbation of a map with a vertical branch to a map with a steep branch, this chaotic repeller persists, becoming non-hidden.

  In the simplest case, the existence of a hidden fixed point and a hidden 2-cycle implies the existence of hidden cycles of all periods, which can be interpreted as an unexpected form of the well-known rule, namely "periods one and two imply chaos" [4].

References

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