Uni-directional wave propagation in time-modulated inerter-based lattice

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Summary. In this work, we investigate the uni-directional wave propagation in elastic locally resonant mechanical metamaterials with inerter elements, where mass and stiffness properties are simultaneously modulated as periodic functions of time. Time-modulated properties of inerter elements are adopted in order to highlight the effect of time-dependent inertia amplification on mass properties. By considering the Bloch theorem and Fourier expansion, the system of Hill equations is reduced to the eigenvalue problem that gives dispersion relations with asymmetric frequency band structures.

Formulation of the problem

Let us consider the mass-spring-inerter systems representing the locally resonant mechanical metamaterial (LRMM) lattice as shown in Figure 1. The model consists of the periodically repeated mass-spring-inerter blocks connected into an infinite chain. The local resonators are represented by small inner masses inside the host outer masses. The local resonators are introduced into the lattice due to the well know properties of generating low-frequency band gaps in the metamaterial structures. Bloch wave propagation analysis is performed for the corresponding unit cell by applying the proper periodic boundary conditions. Figure 1 illustrates the n-th unit cell of the LRMM lattice, which is bounded by red dashed rectangle and it consists of three identical mass-in-mass subsystems connected through time-modulated springs and inerters. Moreover, each mass-in-mass subsystem is connected to the ground through pairs of time-modulated springs and inerters. Note that internal resonators are also connected to the host mass through a time-modulated spring.

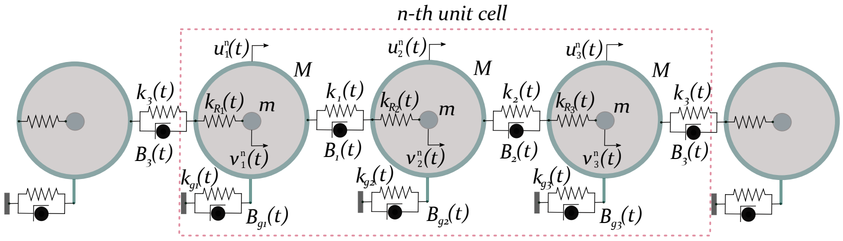


Figure 1: The unit cell of the inerter-based LRMM lattice.

Mathematical model

The governing equations of the locally resonant lattice with time varying stiffness and inerter properties for the n-th unit cell (Figure 1) can be expressed, in a general case, as a system of ordinary differential equations with time dependent coefficients, known as Hill equations [1], as follows

$$M\ddot{u}_r^n + k_{r-1}(t)(u_r^n - u_{r-1}^n) + k_r(t)(u_r^n - u_{r+1}^n) + k_{gr}(t)u_r^n + k_{Rr}(t)(u_r^n - v_r^n) + B_{r-1}(t)(\ddot{u}_r^n - \ddot{u}_{r-1}^n) + \tag{1}$$

$$B_r(t)(\ddot{u}_r^n - \ddot{u}_{r+1}^n) + B_{gr}(t)\ddot{u}_r^n + \dot{B}_{r-1}(t)(\dot{u}_r^n - \dot{u}_{r-1}^n) + \dot{B}_r(t)(\dot{u}_r^n - \dot{u}_{r+1}^n) + \dot{B}_{gr}(t)\dot{u}_r^n = 0,$$

$$m\ddot{v}_r^n + k_{Rr}(t)(v_r^n - u_r^n) = 0, \tag{2}$$

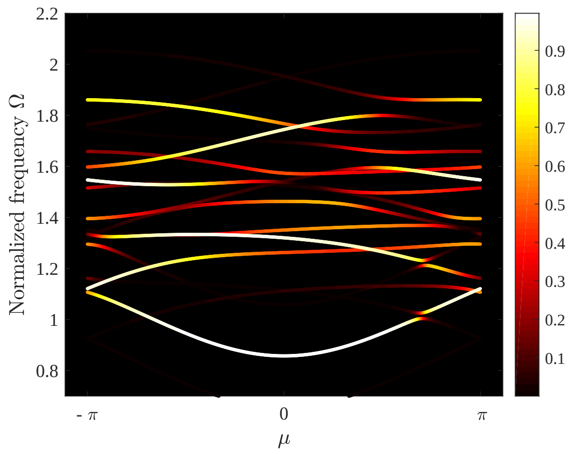
where $r = 1, ..., R$, R is the number of masses in the n-th unit cell.
In order to study the uni-directional wave propagation in the time-modulated one-dimensional lattice structure we limit our analysis to the case when $R = 3$. The differential equation (1) describes the motion of the r-th point mass in the n-th unit cell, and differential equation (2) describes the motion of the internal resonator.

Solution procedures

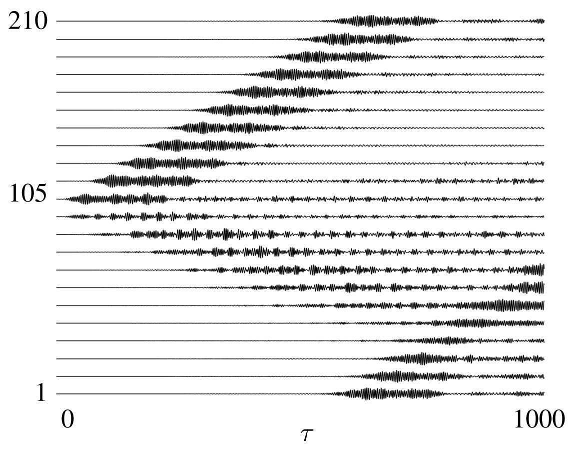
Plane wave method is applied by considering the Fourier series expansion of displacement vector and time-dependent mass and stiffness matrices. We assume that mass and stiffness matrices are modulated with the velocity $v_m = \frac{\lambda_m}{T_m}$. The terms λ_m and T_m are related to the spatial wave length and temporal period of modulation, respectively. By considering the inverse method explained in [2], the plane wave solution of the time-varying lattice model presented in equations (1) and (2) can be assumed in Bloch-Fourier form [3] as

$$\mathbf{u}_n(t) = e^{i(-n\mu+\omega t)} \sum_{p=-\infty}^{\infty} \mathbf{a}_p e^{ip\omega_m t}, \tag{3}$$

where $\mathbf{a}(t) = \mathbf{a}(t + T_m)$ is the vector of periodic amplitude function in time and $\mu = \lambda_m \kappa$ is the non-dimensional wavenumber. Considering the truncated expansion from equation (3), the system of differential equations is reduced to



(a) Fundamental branches of the modulated lattice.



(b) Time response to harmonic excitation of the lattice.

Figure 2: The local resonant structure with time varying inerter and stiffness properties.

the system of $(R+N_r)(2P+1)$ algebraic equations. Therefore, by taking the values of $P=1$ and $Q=2$ in equation (3) the system (1) is reduced to the following quadratic eigenvalue problem

$$(\Omega^2\mathbf{A}_1+\Omega\mathbf{A}_2+\mathbf{A}_3)\mathbf{a}=\mathbf{0}, \tag{4}$$

where $\Omega=\omega/\omega_0$ and $\mathbf{a}$ are eigenvalues and eigenvectors of the system, respectively.

Numerical results

The aim of the numerical study is to verify the applicability of the above proposed method based on the Bloch-Fourier expansion and to perform the dispersion analysis of the time-modulated periodic elastic lattice with local resonators given in the form of one-dimensional chain with inerters (Figure 2a). Moreover, the uni-directional wave propagation is observed based on the solution of the system of equations using the finite difference method for a specified excitation frequency (Figure 2b).

By considering the simultaneous time-modulation of stiffness and inertia properties, the dispersion characteristic of the lattice is presented in Figure 2a. The diagrams show many crossings and folding of dispersion branches within, and at the edges of the first Brillouin zone. The fundamental branches are obtained by using the weighting and threshold process. Moreover, the asymmetric band gaps can be observed for the presented configuration, from which we can find specter of frequencies which can lead to asymmetric or uni-directional wave propagation. Finally, to demonstrate the uni-directional wave propagation we apply the harmonic excitation on the center mass of the finite chain with the picked frequency $\Omega=1.7$. The finite chain has 70 unit cells and Born–von Karman boundary conditions are applied at the ends of the chain. We observe only the displacement amplitudes of outer masses of the chain. The spatial profile in Figure 2b indicates one way traveling of the primary wave in the positive direction $+x$. However, it can be noticed that a small portion of wave energy is traveling in the opposite direction, which can be attributed to the fact that the corresponding asymmetric band gap is influenced by internal resonators of the LRMM lattice.

Conclusions

In this study, we numerically demonstrate how uni-directional wave propagation can occur in time-modulated elastic and locally resonant metamaterial lattices with inerters. We successfully included inerter elements in the lattice structure demonstrating the shift of dispersion frequencies to lower values due to the inertia amplification effect. The Bloch-Fourier based procedure is suggested to analyze the dispersion characteristics of systems with simultaneous time-modulation of stiffness and mass properties. Dispersion characteristic is used to confirm the existence of asymmetric band gaps and then applied as a basis for transient analysis and demonstration of uni-directional wave propagation.

References

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