Recent advances on spectral-submanifold-based model reduction: bifurcations and configuration constraints

Mingwu Li*, Shobhit Jain* and George Haller *
Institute for Mechanical Systems, ETH Zürich, Switzerland

Summary. We show how spectral submanifold theory can be used to construct reduced-order models for forced nonlinear systems possibly with internal resonances. We perform continuation of equilibria (or limit cycles) of the reduced-order models to obtain (quasi-) periodic response of the full system and predict their bifurcations. In addition, we show how to construct reduced-order models for constrained mechanical systems using spectral submanifolds. These reduced-order models enable bifurcation analysis and efficient extraction of backbone and forced response curves of high-dimensional mechanical systems with configuration constraints.

Introduction

The analysis of high-dimensional nonlinear mechanical systems has been a significant challenge. The construction of reduced-order models for original high-dimensional systems is then of great importance because they enable efficient nonlinear analysis. Among these constructions, the reduced-order models based on invariant manifolds are prominent for nonlinear systems as they are supported by rigorous theory. In particular, the theory of *spectral submanifolds* (SSM) has laid a solid foundation for constructing mathematically rigorous reduced-order models [1]. Recent developements [2] have enabled the computation of SSMs and their reduced dynamics in physical coordinates. The software implementation of the method has been available in an open-source package, SSMTOOL [3]. In previous studies, the applications have been limited to two-dimensional SSMs. Here, we derive reduced-order models on higher-dimensional SSMs to analyze systems with internal resonances. We integrate continuation package COCO [4] with SSMTOOL to obtain forced response curves and perform bifurcation analysis of internally resonant mechanical systems. We further demonstrate how SSMTOOL can handle mechanical systems with configuration constraints.

Reduced-order model and bifurcation analysis

Reduced-order model

We consider a periodically forced nonlinear mechanical system

$$\boldsymbol{M}\ddot{\boldsymbol{x}}+\boldsymbol{C}\dot{\boldsymbol{x}}+\boldsymbol{K}\boldsymbol{x}+\boldsymbol{f}(\boldsymbol{x},\dot{\boldsymbol{x}})=\epsilon\boldsymbol{f}^{\mathrm{ext}}(\Omega t), \tag{1}$$

where $\boldsymbol{x}\in\mathbb{R}^n$ is displacement vector; $\boldsymbol{M},\boldsymbol{C},\boldsymbol{K}\in\mathbb{R}^{n\times n}$ are the mass, damping and stiffness matrices; $\boldsymbol{f}(\boldsymbol{x},\dot{\boldsymbol{x}})$ is a C^r smooth nonlinear function such that $\boldsymbol{f}(\boldsymbol{x},\dot{\boldsymbol{x}})\sim\mathcal{O}(|\boldsymbol{x}|^2,|\boldsymbol{x}||\dot{\boldsymbol{x}}|,|\dot{\boldsymbol{x}}|^2)$; and $\epsilon\boldsymbol{f}^{\mathrm{ext}}(\Omega t)$ denotes external harmonic excitation. Let $\boldsymbol{z}=(\boldsymbol{x},\dot{\boldsymbol{x}})$, the equation of motion (1) can be transformed into a first-order form

$$\boldsymbol{B}\dot{\boldsymbol{z}}=\boldsymbol{A}\boldsymbol{z}+\boldsymbol{F}(\boldsymbol{z})+\epsilon\boldsymbol{F}^{\mathrm{ext}}(\Omega t),\quad 0\leq\epsilon\ll 1, \tag{2}$$

with appropriate definition of $\boldsymbol{A},\boldsymbol{B},\boldsymbol{F}$ and $\boldsymbol{F}^{\mathrm{ext}}$. We assume that $\boldsymbol{z}=\boldsymbol{0}$ is an asymptotically stable fixed point at $\epsilon=0$. We construct SSM-based reduced-order model for (2) in following steps

1. We determine the master subspace for reduction based on external and internal resonances. Specifically, we first include the modes subject to (near) external resonance with excitation frequency Ω to the master subspace. We further add the modes having internal resonances with the externally resonant modes to the master subspace.

2. We compute the parametrization $\boldsymbol{W}_\epsilon(\boldsymbol{p},\Omega t)$ of the SSM associated with the master subspace along with its reduced dynamics $\dot{\boldsymbol{p}}=\boldsymbol{R}_\epsilon(\boldsymbol{p},\Omega t)$, where the parameterization coordinates $\boldsymbol{p}$ are of the same dimension as the that of the master subspace. We express the parametrizations $\boldsymbol{W}_\epsilon(\boldsymbol{p},\Omega t)$ and $\boldsymbol{R}_\epsilon(\boldsymbol{p},\Omega t)$ as Taylor series expansion in $\boldsymbol{p}$ and determine the unknown expansion coefficients by balancing polynomials in the invariance equation, as shown in [2].

3. We transform the reduced dynamics $\dot{\boldsymbol{p}}=\boldsymbol{R}_\epsilon(\boldsymbol{p},\Omega t)$ to factor out the Ωt-dependent terms. In particular, using the transformation $\boldsymbol{p}=\boldsymbol{H}(\Omega t)\boldsymbol{q}$ with an explicit diagonal matrix $\boldsymbol{H}$, we obtain an autonomous reduced-order model

$$\dot{\boldsymbol{q}}=\boldsymbol{h}(\boldsymbol{q},\Omega,\epsilon). \tag{3}$$

Further details about the above construction can be found in our preprint [5].

Bifurcation analysis

The bifurcation analysis of the full high-dimensional system (2) is simplified to the bifurcation analysis of the reduced-order model (3). This simplification enables the prediction of bifurcations of periodic/quasi-periodic orbits of the full system (2) via the bifurcations of the equilibria/limit cycles of the reduced-order model (3). For instance, we simply identify Hopf bifurcations of equilibria and limit cycles in the reduced system (3) to predict the existence of two-dimensional and three-dimensional quasi-periodic invariant tori in the full system (2) of arbitrary dimension.

We have integrated the continuation package COCO with SSMTOOL to perform such bifurcation analysis. Specifically, we have developed the following three toolboxes within SSMTOOL.

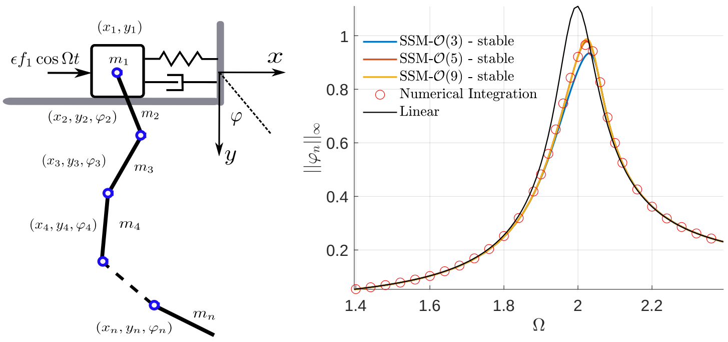

Figure 1: A chain of pendulums (left) with $n = 41$ and the forced response curve of rotation angle of the last pendulum (right).

- `SSM-ep`: continuation of (bifurcated) equilibria of the reduced-order model (3) and the construction of corresponding (bifurcated) periodic orbits of the full system (2);
- `SSM-po`: continuation of (bifurcated) limit cycles of the reduced-order model (3) and the construction of corresponding (bifurcated) two-dimensional quasi-periodic invariant tori of the full system (2);
- `SSM-tor`: continuation of two-dimensional quasi-periodic invariant tori of the reduced-order model (3) and the construction of corresponding three-dimensional quasi-periodic invariant tori of the full system (2).

More details about these toolboxes and their applications to finite-element examples can be found in our preprints [5, 6].

Extension to constrained mechanical systems

Now, we consider a periodically forced nonlinear mechanical system with configuration constraints of the form

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} + \boldsymbol{f}(\boldsymbol{x}, \dot{\boldsymbol{x}}) + \boldsymbol{G}^\top\boldsymbol{\lambda} = \epsilon\boldsymbol{f}^{\mathrm{ext}}(\Omega t), \quad \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{0} \tag{4}$$

where $\boldsymbol{x}, \boldsymbol{M}, \boldsymbol{C}, \boldsymbol{K}, \boldsymbol{f}, \epsilon, \boldsymbol{f}^{\mathrm{ext}}$ carry the same definitions as in (1), $\boldsymbol{g} : \mathbb{R}^n \to \mathbb{R}^m$ $(m < n)$ represents configuration constraints, $\boldsymbol{G} = \partial\boldsymbol{g}/\partial\boldsymbol{q}$, and $\boldsymbol{\lambda}$ denotes the Lagrange multipliers corresponding to the configuration constraints.
With $\boldsymbol{z} = (\boldsymbol{x}, \dot{\boldsymbol{x}}, \boldsymbol{\lambda})$, we can again transform (4) into the first-order form (2) with appropriate definition of $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{F}$ and $\boldsymbol{F}^{\mathrm{ext}}$. We assume $\boldsymbol{g}(\boldsymbol{0}) = \boldsymbol{0}$ such that the origin $\boldsymbol{z} = \boldsymbol{0}$ is still a fixed point of the full system (2). We further assume that $\boldsymbol{G}(\boldsymbol{0})$ is of full rank, i.e., the constraints $\boldsymbol{g}$ are not redundant and the origin is not a singular configuration. As a result, the vector $\boldsymbol{\lambda}$ of the Lagranian multipliers is well-defined. Under these assumptions, the matrix $\boldsymbol{B}$ is singular and the matrix pair $(\boldsymbol{A}, \boldsymbol{B})$ has $3m$ infinite eigenvalues corresponding to the m configuration constraints.
Indeed, the computation of SSM and its associated reduced-order model does not require the invertibility of $\boldsymbol{B}$ matrix [2]. This enables us to use SSMTOOL to construct SSM-based reduced-order model for the full system (4) following the same procedure as in the previous section. Hence, we can again perform bifurcation analysis and obtain backbone and forced response curves of the constrained system (4) directly from the SSM-based reduced-order (3).
As a demonstration, we consider a chain of planar pendulums attached to a harmonically excited oscillator, as shown in Fig. 1. The configuration constraints in this system come from the revolute joints. We extract the forced response curve of the system via a two-dimensional SSM-based reduced-order model. As shown in the right panel of Fig. 1, the forced response curve predicted by this reduced-order model agrees with the one obtained from the numerical time integration of the full system, thereby demonstrating the effectiveness of SSM reduction.

Conclusions

We derived SSM-based reduced-order models for nonlinear systems with internal resonances. Such reduced-order models enable efficient nonlinear analysis of the full systems with arbitrary dimensions. We further extended this analysis to constrained mechanical systems where the equations of motion are in the form of differential-algebraic equations.

References

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