Coupled nonlinear oscillators: closed form solutions

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Summary. The mathematical solution of a system of two coupled Duffing oscillators is obtained in closed form by extending to this 2 DOF system a technique previously used for 1 DOF systems. A parametric investigation is proposed to illustrate the usefulness of the exact solution to detect the main dynamical phenomena, in particular those related to the modal coupling which is the main characteristic of this archetypal system.

Introduction

A two Degrees of Freedom (DOF) nonlinear oscillator is an archetypal system that allow to detect, in a simple way, the modal coupling in nonlinear mechanical systems. It permits focusing on the main phenomena, without unessential mathematical developments that sometimes hide the main physical characteristics of interest. Furthermore, it is obtained when a two-mode reduced order model of any, even infinite dimensional, system is considered. It is the natural evolution of the study of simple 1 DOF archetypal systems, like Duffing, Helmholtz, van der Pool, etc., that have been largely studied in the past.
Although it looks a simple model, it is not yet fully investigated, even if several studies have been devoted to modal coupling of different, specific, engineering systems. Furthermore, all previous works used numerical simulations or approximated analytical methods, notably the multiple time scale method.
In this work, which is based on [1] and take advantage from the ideas of [2-4], we consider an exact, closed form, solution for a system of two Duffing oscillators, linearly and nonlinearly coupled. This permits a full parametric investigation and detection of "all" dynamical outcomes due to coupling.

Governing equations and solution

A system of two Duffing linearly and nonlinearly coupled oscillators is governed by the equations

$$M_x\ddot{x} + D_x\dot{x} + K_x x + K_{3x}x^3 + D_{xy}(\dot{x} - \dot{y}) + K_{xy}(x - y) + K_{3xy}(x - y)^3 = G_x(t),$$
$$M_y\ddot{y} + D_y\dot{y} + K_y y + K_{3y}y^3 + D_{xy}(\dot{y} - \dot{x}) + K_{xy}(y - x) + K_{3xy}(y - x)^3 = G_y(t), \quad (1)$$

where M_i are the masses, D_i the damping coefficients, K_i the linear stiffnesses, K_{3i} the nonlinear stiffnesses and $G_i(t)$ the external forces. To obtain the closed form solution, the excitations are assumed to be in the form

$$G_x(t) = D_x\dot{x} + D_{xy}(\dot{x} - \dot{y}) + K_{xy}(x - y) + K_{3xy}(x - y)^3 - S_x x - S_{3x}x^3,$$
$$G_y(t) = D_y\dot{y} + D_{xy}(\dot{y} - \dot{x}) + K_{xy}(y - x) + K_{3xy}(y - x)^3 - S_y y - S_{3y}y^3, \quad (2)$$

where S_x, S_{3x}, S_y, S_{3y} are parameters that can be chosen freely. Inserting (2) in (1) yields

$$\ddot{x} + (\omega_x^2 + W_x)x + (k_x + C_x)x^3 = 0, \quad \ddot{y} + (\omega_y^2 + W_y)y + (k_y + C_y)y^3 = 0, \quad (3)$$

where

$$\omega_x^2 = \frac{K_x}{M_x}, \ W_x = \frac{S_x}{M_x}, \ k_x = \frac{K_{3x}}{M_x}, \ C_x = \frac{S_{3x}}{M_x}, \ \omega_y^2 = \frac{K_y}{M_y}, \ W_y = \frac{S_y}{M_y}, \ k_y = \frac{K_{3y}}{M_y}, \ C_y = \frac{S_{3y}}{M_y}. \quad (4)$$

Equations (3) are two uncoupled Duffing equations, for which the closed form solutions are

$$x(t) = A_x cn(a_x t, b_x), \quad y(t) = A_y cn(a_y t, b_y), \quad (5)$$

where

$$a_x^2 = (\omega_x^2 + W_x) + (k_x + C_x)A_x^2, \quad b_x^2 = \frac{(k_x + C_x)A_x^2}{2a_x^2},$$
$$a_y^2 = (\omega_y^2 + W_y) + (k_y + C_y)A_y^2, \quad b_y^2 = \frac{(ky_x + C_y)A_y^2}{2a_y^2}. \quad (6)$$

and where "cn" is the Jacobian elliptic function. $x(t)$ and $y(t)$ are periodic with period

$$T_x = \frac{4K(b_x)}{a_x}, \quad T_y = \frac{4K(b_y)}{a_y}. \quad (7)$$

The solutions of interest are those for which $T_x = T_y(= T)$, i.e. both $x(t)$ and $y(t)$ oscillate with the same period (but not with the same amplitudes A_x and A_y). Actually, this is an equation linking A_x and A_y (e.g. $A_y(A_x)$), once all the other parameters are known. Then, from (7) one gets the period of the excitation $T_x = T(A_x)$ and $T_y = T(A_y)$. Inverting these expressions one obtains the frequency response curves $A_x(T)$ and $A_y(T)$. Further details, including how to determine its stability, can be found in [1].
An advantage of the proposed method is that it is possible to use the free parameters S_x, S_{3x}, S_y, S_{3y} to shape the excitation and to have it as close as possible to a desired target, still keeping the closed form expressions for the *nonlinear* oscillations.

An example

To shortly illustrate the previous findings we consider

$$\omega_x = 2.5,\ \ W_x = -4.5,\ \ k_x = 2,\ \ C_x = 0,\ \ \omega_y = 2.5,\ \ W_y = 0,\ \ k_y = -1,\ \ C_y = 0,$$
$$M_x = 1,\ \ M_y = 2,\ \ D_x = 0.01,\ \ D_{xy} = 0.02,\ \ D_x = 0.03,\ \ K_{xy} = 0.5,\ \ K_{3xy} = 0.4, \tag{8}$$

which corresponds to the perfect internal resonance ($\omega_x = \omega_y$) with the x mode hardening and the y mode softening. The solution of $T_x = T_y$ is reported in Fig. 1. For $A_x = 1$ one obtains $A_y = 1.9731$ and $T = 3.5009$. The corresponding excitations $G_x(t)$ and $G_y(t)$ and solutions $x(t)$ and $y(t)$ are illustrated in Fig. 2. In Fig. 3 it is shown how choosing $W_x = -4.47979443$, $C_x = 0.036619325$, $W_y = 0.14970754$, $C_y = 0.024427757$ allows to strongly reduce $G_y(t)$, by leaving practically unchanged $G_x(t)$. This is an example of shaping the excitation, which can be improved by using optimization algorithms. Much more results, including frequency response curves, are reported in [1].

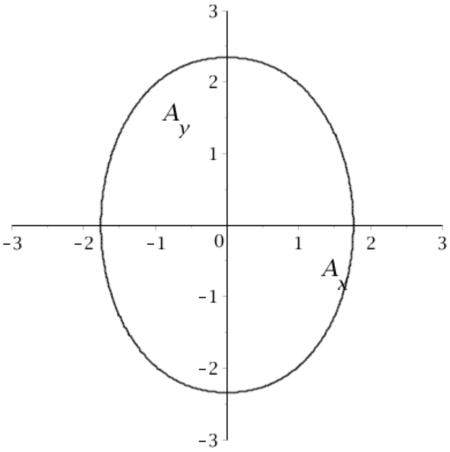

Figure 1: The solution $A_y(A_x)$ for the parameter (8).

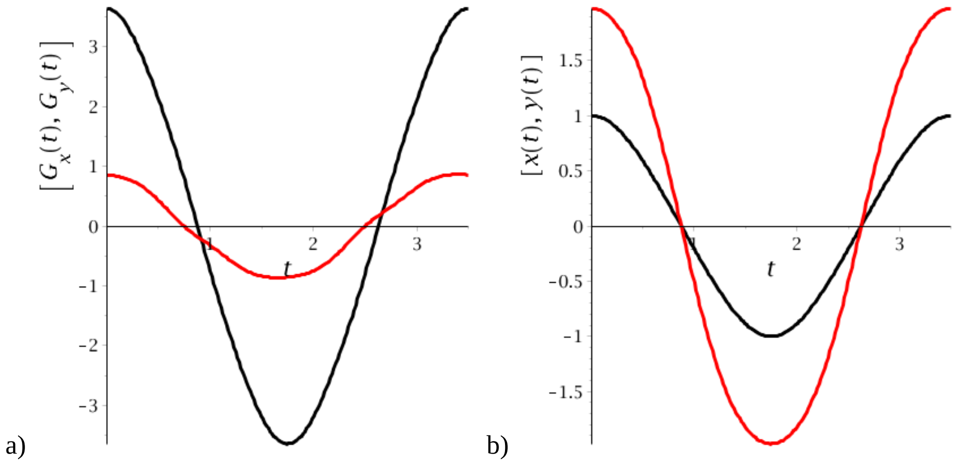

Figure 2: a) $G_x(t)$ (black) and $G_y(t)$ (red); b) $x(t)$ (black) and $y(t)$ (red). $A_x = 1$ and parameters (8).

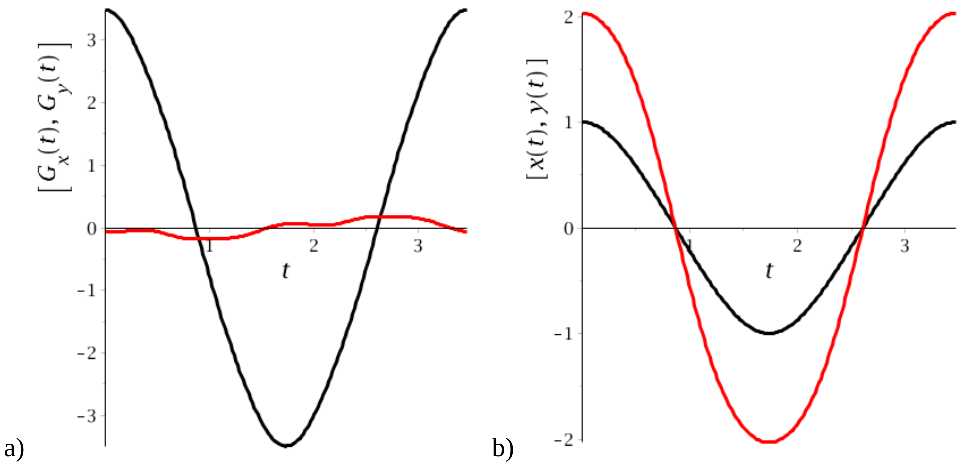

Figure 3: As Fig. 2, but with $W_x = -4.47979443$, $C_x = 0.036619325$, $W_y = 0.14970754$, $C_y = 0.024427757$.

References

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