Experimentally validated geometrically exact model for nonlinear dynamic analysis of cantilevers undergoing extreme motions

Hamed Farokhi*, Yiwei Xia† and Alper Erturk†
*Department of Mechanical and Construction Engineering, Northumbria University, Newcastle upon Tyne NE1 8ST, UK
†George W. Woodruff School of Mechanical Engineering, Georgia Institute of Technology, Atlanta, 30332, USA

Summary. A unique characteristics of cantilevered beams is their capability of undergoing very large-amplitude oscillations, which has been used in various applications such as energy harvesting and bio-inspired actuation. To examine cantilever motions of extremely large amplitudes, the third-order inextensible nonlinear model cannot be used, hence requiring the application of a geometrically exact model. This study presents an experimental and theoretical investigation on nonlinear extreme dynamics of cantilevers under base excitation. A rotation-based geometrically exact model is developed using Euler-Bernoulli beam theory and inextensibility assumption to examine the cantilever response at extreme motion amplitudes. Precise experiments are conducted using a state-of-the-art in vacuo base excitation experimental set-up to drive the cantilever to extremely large motion amplitudes, and a high-speed camera is used to capture the motion. Extensive comparisons are conducted between experimental and theoretical results and it is shown that the proposed exact model can be used reliably to capture cantilever motions of extreme amplitudes.

Introduction

Cantilevered beams are present in different engineering applications such as vibration energy harvesters, micro gyroscopes, and piezoelectric sensors and actuators. Owing to have one end free, they can undergo large-amplitude vibrations. However, analysing large-amplitude vibration is a difficult task, not only due to the presence of different sources of nonlinearity, but also the fact that the amplitude could grow extremely large, rendering nonlinear truncated models inaccurate and requiring a geometrically exact model. Many investigators have examined the vibrations of cantilevers over the last few decades, starting with the well-known third-order model developed by Crespo da Silva and Glynn [1, 2], which was later further examined by Nayfeh and Pai [3, 4]. Many other studies utilised the third-order model to examine the behaviour of cantilevers, namely Dwivedy and Kar [5] and Mahmoodi et al. [6]. Farokhi et al. [7] continued the studies on this topic by developing a dynamical version of the rotation-based exact nonlinear cantilever model, originally used for static buckling of an elastic continuum [8], capable of capturing motions of extreme amplitudes. This study reports detailed experimental results on extremely large vibrations of a cantilever and shows comparisons between experimental and theoretical results.

Model Development

To derive the rotation-based geometrically exact equation of motion, a cantilevered beam is considered of length L, cross-sectional area A, second moment of area I, Young's modulus E, mass per unit length m, and material damping coefficient η, under harmonic excitation in the form of $z_0 \sin(\omega_z t)$. Making use of the inextensibility assumption, and in the context of the Euler-Bernoulli beam theory, the longitudinal (u) and transverse (w) displacements of the cantilever (the latter with respect ot the clamped end), can be formulated in terms of the centreline rotation ψ as

$$u(s,t) = \int_0^s \Big(\cos\psi(\xi,t) - 1\Big)\,\mathrm{d}\xi, \qquad w(s,t) \;= \int_0^s \sin\psi(\xi,t)\,\mathrm{d}\xi. \tag{1}$$

Assuming a vertical configuration for the cantilever with the clamped end at the bottom (see Fig. 1(a)) and taking into account the gravity, the geometrically exact equation of motion for centreline rotation of the cantilever can be derived as

$$\begin{aligned}
&J\partial_{tt}\psi + m\sin\psi \int_s^L \int_0^{s^*} \Big(\partial_{tt}\psi(\xi,t)\sin\psi(\xi,t) + \big(\partial_t\psi(\xi,t)\big)^2\cos\psi(\xi,t)\Big)\,\mathrm{d}\xi\,\mathrm{d}s^* \\
&+ m\cos\psi \int_s^L \Big[- z_0\omega_z^2\sin(\omega_z t) + \int_0^{s^*} \Big(\partial_{tt}\psi(\xi,t)\cos\psi(\xi,t) - \big(\partial_t\psi(\xi,t)\big)^2\sin\psi(\xi,t)\Big)\,\mathrm{d}\xi\Big]\,\mathrm{d}s^* \\
&- EI\partial_{ss}\psi - \eta I\partial_{tss}\psi - mg(L-s)\sin\psi = 0.
\end{aligned} \tag{2}$$

in which $J = \rho I$ with ρ being the mass density, g is the acceleration due to gravity, and $\partial_s \equiv \partial/\partial s$. Equation 2 is then nondimensionalised and discretised into a 6-degree-of-freedom system using the Galerkin technique while keeping all terms geometrically exact. The resultant set is solved using a continuation technique.

Results

Figure 1(b) shows the in vacuo base excitation experimental set-up used to excite the cantilever in the primary resonance region and to drive it to extremely large amplitude. A high-speed camera system was used to capture vibration response of the cantilever. An image processing code was then developed to extract the beam deformed shape and the motion amplitudes from the videos. In order to examine oscillations of extremely large amplitudes, the cantilever was excited

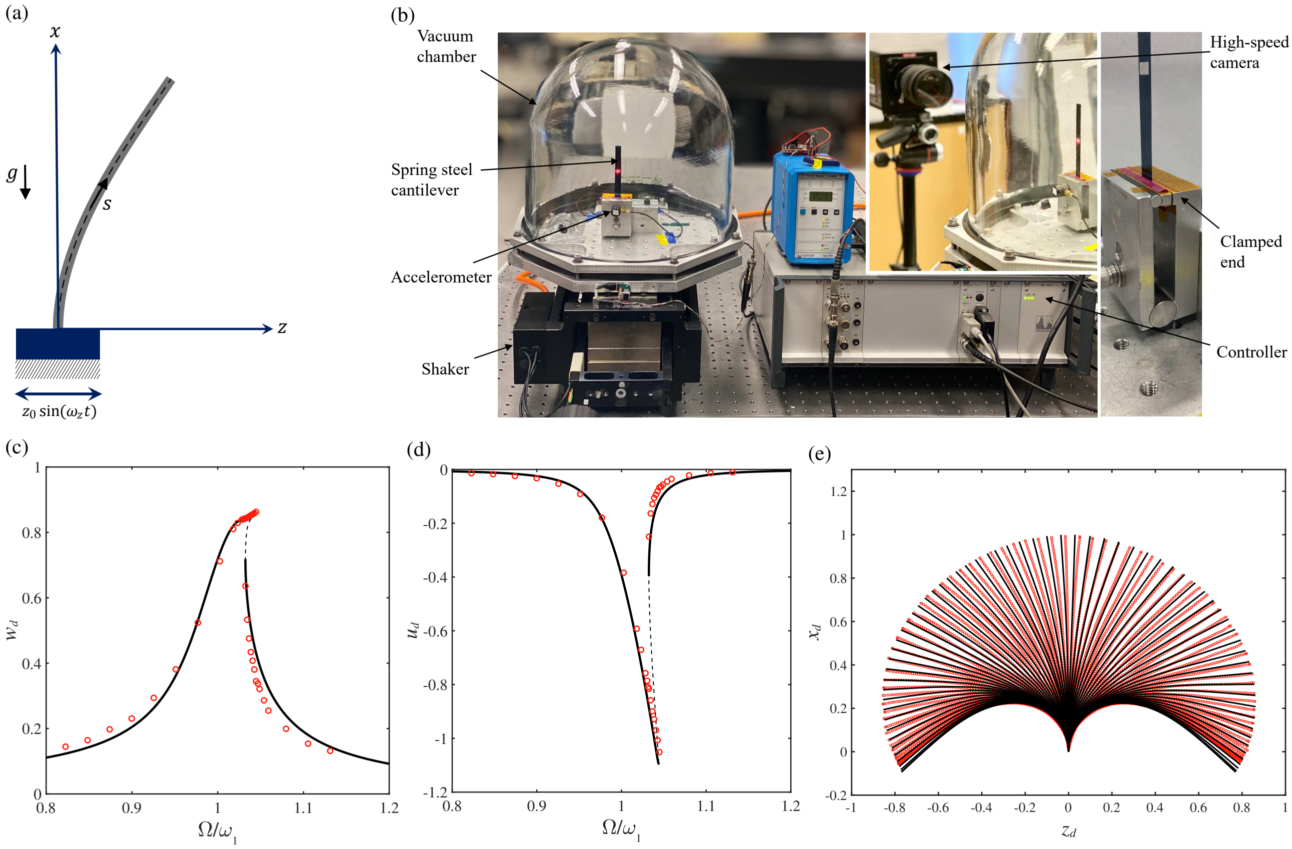

Figure 1: (a) Schematic of the cantilever. (b) In vacuo base excitation experimental set-up. (c, d) Frequency response of the cantilever tip transverse and longitudinal vibration at 0.5g RMS acceleration level, respectively. (e) Motion of the cantilever at peak amplitude in one period of oscillation. In sub-figure (c)-(e), circles denote experimental results and lines show geometrically exact model predictions.

at base acceleration of $0.5g$ RMS under near-vacuum conditions (i.e. 9% atmospheric pressure). The theoretical and experimental frequency responses of the cantilever are illustrated in Fig. 1(c) and 1(d), and the oscillation of the cantilever at its peak resonance amplitude in one period of oscillation is shown in Fig. 1(e). It is seen in Fig. 1(c, d) that the exact model works very well even at this extreme oscillation amplitude and predicts a frequency response amplitude very close to the experimental one. Additionally, as shown in Fig. 1(e), the exact model does an excellent job at capturing the extreme vibrations of the cantilever, even when the tip of the cantilever bends backwards, as confirmed by the experimental observations. Hence, these results prove the validity of the proposed geometrically exact model and that it can be used reliably for studying cantilever vibrations of extremely large amplitudes.

Conclusions

Nonlinear extreme dynamics of a cantilever was examined experimentally and theoretically. For the experimental part, an in vacuo experimental set-up was used to excited the cantilever in the primary resonance region and to drive the cantilever vibration to extremely large amplitudes, and for the theoretical part, a geometrically exact model based on Euler-Bernoulli beam theory was developed. Comparisons between experimental and theoretical results showed that the proposed exact model is fully capable of capturing vibrations of extremely large amplitudes reliably.

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