Influence of vaccination and social distancing on epidemic prevention

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Summary. To analyse the effect of vaccination strategies and the reinfection or temporary acquired immunity of individuals in a population with virus presence as well the social distancing, a variation of the SIR (Susceptible-Infected-Removable) model is proposed. The calculation of the equilibrium points for the stability analysis of the system is performed. Two equilibrium points were found, one disease-free and the other endemic, for which the existence conditions are discussed. The stability of the points was analysed and the results were verified through simulations varying the parameters of vaccination, reinfection and social distancing.

Introduction

Due to technological advances, which used to take time to get from one place to another, today it travels in a matter of hours, facilitating the risk of the appearance of a new virus quickly turning into a pandemic. The interest in studying the modeling of infectious diseases lies in understanding the mechanisms of transmission and thus being able to establish prevention policies. Mathematical modelling is useful to show the dynamics of disease spread and indicate which parameters are relevant to guide control strategies.

One of the ways to study the spread of a disease is to establish models composed of a set of formal mathematical symbols to relate population groups, dividing them into compartments, giving rise to compartmental models [1, 2]. These models are an approximation of the real relationships existing in the object of study [3, 4]. For this matter, differential equations are used that establish dynamic relationships between their states depending on the rate of infection, social isolation, mortality, recovery, and vaccination rate [5].

These models contributed to the study of Covid-19 [6]. A disease that quickly became a serious pandemic, claiming the lives of millions of people around the world [7]. The present work aims to analyse the influence of the vaccination rate, as well as the influence of its effectiveness in reducing the spread of the virus.

For this, it is proposed a modification in the Susceptible - Infected - Removed (SIR) compartment model proposed by Kermack and McKendrick in 1927 [8, 9, 10]. In this new model, social distancing, the effect of the vaccine [11, 12], and its effectiveness in coordinated actions are considered. We want to study the influence of varying the effectiveness of vaccination on the endemic balance [13, 14, 15], as well as the effort needed to ensure its stability.

In addition, it is intended to evaluate the influence of the model parameters on the basic reproduction number of the infection (R_0). Which measures the infectivity of a pathogen in an environment in which no one has acquired immunity to it. With this parameter indicate the effective reproduction number (R_e), exposed to the real conditions of disease evolution and relating to the influence of variations in efficacy on vaccination strategies. The objective of the article is to study the effects of vaccination to prevent the spread of epidemics, to determine the minimum effort of the vaccine as well as the effect of reinfection in the disease control process.

Models Descriptions

The proposed model is a modification of the original SIR model proposed by Kermack and Mckendrick [8, 9, 10]. In this model, the susceptible population S is infected at a rate when it comes in contact with an infected individual from I. The effect of social distancing measures in the susceptible individuals is introduced by the parameter θ, and the subject to the condition $0 < \theta < 1$ and ω is the group to which vaccination is given.

The compartment I represents the infectious population in the incubation phase prior to the onset of symptoms and this population can be asymptomatic or symptomatic. The total population is considered constant, the mortality rate is equal for members of all classes, μ is the mortality and birth rate assumed to be equal, β is the recovery rate, and δ is the reinfection rate, as shown in figure 1.

The model assumes the following hypotheses:

- Fixed population;
- The ways to stop being susceptible is if a person becomes infected, if he is immunized by vaccination or by the mortality rate;
- When the person recovers, they receive permanent immunity;
- The probability of infection is not affected by age, sex or social status;

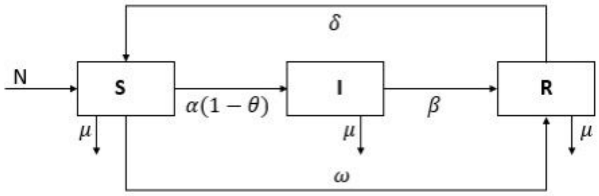

Figure 1: SIR model with vaccine influence, social isolation and reinfection rate.

- The birth and death rate are part of the considerations;
- All births fall into the susceptible class;
- The mortality rate is the same for all compartments and mortality is assumed to be equal to the birth rate.
- The reinfection rate demonstrates the possibility that the individual will be susceptible again.

The model assumes the following notations:
S(t) Number of susceptible individuals at time (t); I(t) Number of infected individuals at time (t); R(t) Number of individuals recovered at time (t); α probability of a susceptible individual becoming infected; β: probability of an infected recovering; θ: social isolation rate; ω vaccination rate of the susceptible; μ mortality rate; δ reinfection rate; N the death is equal for members of all three classes, and it is assumed that the birth and death rates are equal so that the total population is stationary.

Equations
Considering these elements, the model can be described as:

$$\begin{cases} \dot{S} & = & \mu N - \frac{\alpha(1-\theta)S(t)I(t)}{N} - \omega S(t) - \mu S(t) + \delta R(t); \\ \dot{I} & = & \frac{\alpha(1-\theta)S(t)I(t)}{N} - \beta I(t) - \mu I(t); \\ \dot{R} & = & \beta I(t) + \omega S(t) - \mu R(t) - \delta R(t). \end{cases} \tag{1}$$

With constant populations:

$$\dot{S} + \dot{I} + \dot{R} = 0. \tag{2}$$

consequently:

$$S(t) + I(t) + R(t) = N. \tag{3}$$

Taking into account the population density:

$$s = \frac{S}{N}; i = \frac{I}{N}; r = \frac{R}{N}. \tag{4}$$

By substituting (4) in (1):

$$\begin{cases} \dot{s} & = & \mu - \alpha(1-\theta)si - \omega s - \mu s + \delta r; \\ \dot{i} & = & \alpha(1-\theta)si - \beta i - \mu i; \\ \dot{r} & = & \beta i + \omega s - \mu r - \delta r; \end{cases} \tag{5}$$

with the initial conditions $s(0) \geq 0$, $i(0) \geq 0$ and $r(0) \geq 0$.
Here μ is the recruitment and natural death rate, α is the effective contact rate between susceptible and infected individuals, ω is the rate of vaccination, θ is the social isolation and δ is the reinfection rate. All the parameters are positive and for θ the restriction considered $0 < \theta < 1$.

Disease-free and endemic equilibrium points
To investigate the influence of the introduction of feedback from the recovered individuals with no immunity, the equilibrium points from (5) must be determined and their stability must be discussed.
For the proposed model, there are two equilibrium points: one endemic and the other free from infection.
Disease-free equilibrium point:

- P_1 (s^*,i^*,r^*) = ($\frac{\mu+\delta}{\omega+\mu+\delta}$, 0, $\frac{\omega}{\omega+\mu+\delta}$);

Endemic equilibrium point:

- P_2 (s^*,i^*,r^*) such as:
- s^* = $(\frac{\beta+\mu}{\alpha(1-\theta)})$;
- i^* =$(\frac{\mu(\mu+\delta)(\alpha(1-\theta))+\delta\omega(\beta+\mu)-(\beta+\mu)(\omega+\mu)(\mu+\delta)}{(\alpha(1-\theta))\mu(\beta+\mu+\delta)})$;
- r^* = $(\frac{\beta}{\mu+\delta}i^* + \frac{\omega(\beta+\mu)}{(\mu+\delta)(\alpha(1-\theta}$);

Consequently, the existence condition for the endemic equilibrium P_2 is given by

$$\mu(\mu+\delta)(\alpha(1-\theta)) + \delta\omega(\beta+\mu) > (\beta+\mu)(\omega+\mu)(\mu+\delta). \tag{6}$$

and obtain:

$$\omega < \mu\left(\frac{\alpha(1-\theta)}{(\beta+\mu)} - 1\right) \tag{7}$$

From the analysis of the endemic point, it can be seen that to guarantee its existence, it is necessary to respect the condition 7. Analyzing this point, the minimum vaccination effort necessary to reach the point free of infection can be concluded.

Stability analysis

In order to analyse the local stability of the system, the jacobian of the model is calculated at the equilibrium points.

$$J = \begin{bmatrix} -\alpha(1-\theta)i^* - \omega - \mu & -\alpha(1-\theta)s^* & \delta \\ \alpha(1-\theta)i^* & \alpha(1-\theta)s^* - \beta - \mu & 0 \\ \omega & \beta & -\delta-\mu \end{bmatrix}.$$

Analyzing the Jacobian at point P_1:

$$J_{P1} = \begin{bmatrix} -\omega-\mu & -\alpha(1-\theta)s^* & 0 \\ 0 & \alpha(1-\theta)s^* - \beta - \mu & 0 \\ \omega & \beta & -\mu \end{bmatrix}.$$

Using the mathematical tool Matlab 2015, the eigenvalues of the resulting Jacobian matrix are calculated, in order to analyze the stability of the equilibrium point.
Eigenvalues P_1:
$\lambda_1 = -\mu$;
$\lambda_2 = -\omega - \mu - \delta$;
$\lambda_3 = \alpha(1-\theta)s^* - \beta - \mu$.
The stability analysis for model figure 1 presents the disease free equilibrium point and, considering the existence condition, the eigenvalues are given by: $\lambda_1 = -\mu$, $\lambda_2 = -\omega-\mu-\delta$ and $\lambda_3 = \alpha(1-\theta)s^* - \beta - \mu$. The third eigenvalue indicates that if ($s^* < \frac{\beta+\mu}{\alpha(1-\theta)}$) the system is asymptotically stable and if ($s^* > \frac{\beta+\mu}{\alpha(1-\theta)}$) the system becomes unstable indicating a bifurcation in the parameter space.
substituting the variable s^* for the expression calculated at the disease-free equilibrium point, we obtain:

$$\omega > \frac{(\mu+\delta)(\alpha(1-\theta))}{\beta+\mu} - \mu - \delta, \tag{8}$$

from 8 we can conclude the minimum necessary effort of the vaccination strategy to eradicate the epidemic.

Numerical experiments

In order to clarify the stability results obtained analytically for points P_1 and P_2, a series of numerical experiments were carried out.
From the analysis of point P_1 the bifurcation condition 8 is obtained for which the system can behave in a stable or unstable way. To show this behaviour, simulations were performed by varying the values of the system parameters. Figure 2, figure 3 show that for any initial condition, point P_1 it's stable if $\omega > \frac{(\mu+\delta)(\alpha(1-\theta))}{\beta+\mu} - \mu - \delta$.
On the other hand, when the condition $\omega < \frac{(\mu+\delta)(\alpha(1-\theta))}{\beta+\mu} - \mu - \delta$ is set, the system always reaches stability at the endemic point P_2 , behaviour that is shown in figures 5, 6 and 7.
Figure 4 shows that starting from an initial condition (S,I,R)=(0.9, 0.12, 0.08), close to the equilibrium point P_1, and with parameter values $\alpha = 0.6, \beta = 0.6, \theta = 0.1, \delta = 0.2, \mu = 0.01$ and $\omega = 0.3$. guaranteeing the stability condition, the P_1 point is reached.
Figure 5 shows that starting from an initial condition (S,I,R)=(0.3, 0.7, 0.0), far to the equilibrium point P_1, and with parameter values $\alpha = 0.6, \beta = 0.6, \theta = 0.1, \delta = 0.2, \mu = 0.01$ and $\omega = 0.3$. guaranteeing the stability condition, the P_1 point is reached.

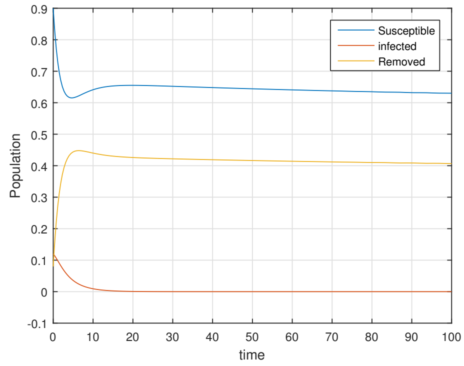

Figure 2: disease free point.

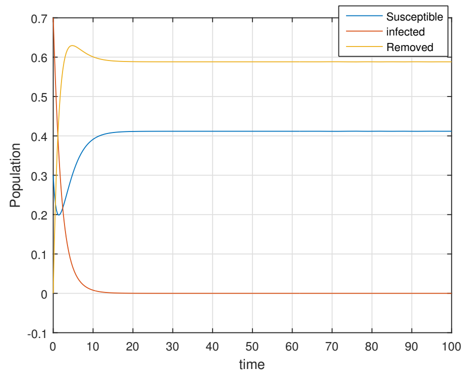

Figure 3: disease free point.

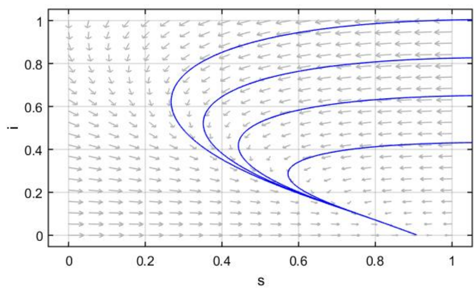

Figure 4: disease free point phase.

In the figure 6 shows the phase diagram of the system with the combination of parameters so that the existence of a disease-free equilibrium point is guaranteed. Each trajectory represents a possible initial condition of the system variables (Susceptible and Infected) and their evolution over time until reaching the equilibrium point. The arrows indicate the direction of movement of the trajectories.

Figure 7 shows that starting from an initial condition (S,I,R)=(0.3, 0.7, 0.0), close to the equilibrium point P_2, and with parameter values $\alpha = 0.9, \beta = 0.3, \theta = 0.1, \delta = 0.2, \mu = 0.3$ and $\omega = 0.1$. guaranteeing the stability condition, the P_2 point is reached.

Figure 8 shows that starting from an initial condition (S,I,R)=(0.8, 0.2, 0.0), far from the equilibrium point P_2, and with parameter values $\alpha = 0.9, \beta = 0.3, \theta = 0.1, \delta = 0.2, \mu = 0.3$ and $\omega = 0.1$. guaranteeing the stability condition, the P_2 point is reached.

In the figure 9 shows the phase diagram of the system with the combination of parameters so that the existence of a endemic equilibrium point is guaranteed. Each trajectory represents a possible initial condition of the system variables (Susceptible and Infected) and their evolution over time until reaching the equilibrium point. The arrows indicate the

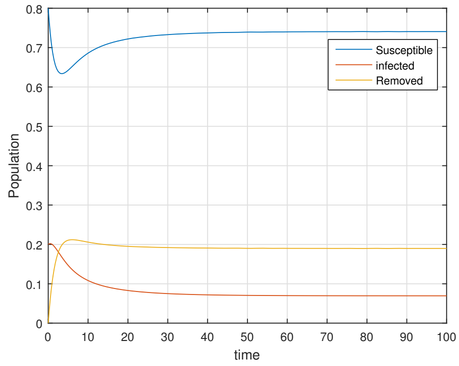

Figure 5: Endemic point

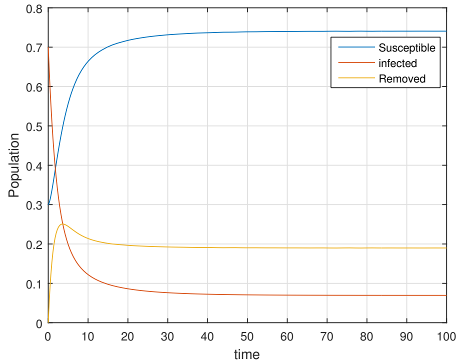

Figure 6: Endemic point

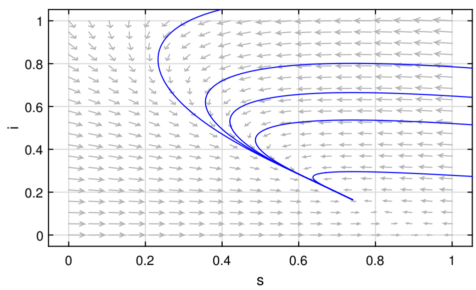

Figure 7: Endemic point phase

direction of movement of the trajectories.
The proposed model is intended to show that in a population where the disease has a temporary immunity character, there is a minimum vaccination effort necessary to eradicate the disease. This result can be verified through variations in the change in the parameters of vaccination, reinfection and social distancing.
In order for the disease-free point to be reached, a minimum vaccination effort given by the equation 8. If the vaccination strategy does not follow this regime, the disease will remain in the population indicating endemic characteristics.

Conclusion

This paper presented a modification of the Susceptible-Infected-Removed (SIR) compartmental model proposed by Kermack and McKendrick in 1927. The social distance, the effect of the vaccine, and its effectiveness in coordinated actions were considered. In addition, the influence of variation in vaccination efficacy on the endemic equilibrium, and the effort required to ensure its stability, were studied.

From the analytical and numerical results, it can be said that the model has two equilibrium points, one endemic and the other disease-free, the existence of each being given by a bifurcation condition that depends on the probability of infection, the social distance and the recovery and infection rates. From there it is possible to find the minimum effort necessary to prevent the epidemic from occurring.
As shown above the existence of the endemic or disease-free point depends on the value of the existence condition. The higher the vaccination and social distancing, the faster the disease-free point will be reached.

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