# Shape Optimization of Curved Mechanical Beams for Zero-Dispersion Point

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*Summary*. In this study, we develop an optimization procedure of zero-dispersion point in curved mechanical beams. The zero-dispersion point is associated with a zero-slope in the frequency-amplitude relation of a nonlinear resonator. As an outcome, local to this zero-dispersion point, the nonlinear effect of amplitude-to-frequency noise conversion is eliminated, albeit the large oscillation amplitude. This zero-dispersion point can be used for noise suppression and frequency stabilization of precision clocks and sensitive detectors. The overall goal is to obtain the zero-dispersion point at the highest possible amplitude (for signal-to-noise ratio enhancement) and frequency (for resolution enhancement). To this end, we found the optimal midpoint elevation of the curved beam and optimized its initial bell-shape function using a genetic algorithm to maximize the frequency and energy level of the zero-dispersion point.

Many technological applications use micromechanical beams as a frequency-selective element to attain high RF frequencies. The practical requirement to operate above the noise floor of the device necessitates the need for large-amplitude oscillation (relative to the small size of the micro-beam); these large amplitudes lie deep in the nonlinear range, where there is considerable amplitude-to-frequency (A-f) noise conversion. However, in a specific class of nonlinear resonators, the dependence of oscillation frequency on the energy (or amplitude) may be non-monotonic, generating energy levels where, locally, the frequency is independent of the energy. For example, the oscillation frequency $\omega(E_{\rm tot})$ can exhibit a softening behaviour for low energy levels $E_{\rm tot} < E_{\rm ZD}$, and hardening behaviour for high energy levels $E_{\rm tot} > E_{\rm ZD}$, where in the transition between these two behaviours, there is an extremum of the frequency corresponding to a zero-dispersion (ZD) point $E_{\rm tot} = E_{\rm ZD}$ satisfying the condition $d\omega/dE_{\rm tot}|_{E_{\rm ZD}} = 0$ (Fig. 1, left panel). Hence, small amplitude fluctuations are not translated into frequency fluctuations near this ZD point. Consequently, the ZD point can be used for suppression of frequency noise in both open-loop [1] and closed-loop [2] systems at large vibration amplitudes, which guarantee operation above the noise floor of the device with a large signal-to-noise ratio (SNR).
In this study, we analyze the possibility to generate an optimal ZD point from the well-known and thoroughly explored curved micro-beam [3, 4] (Fig. 1, right panel). In the curved beam, there are inherently hardening and softening nonlinearities, and therefore, we can obtain the ZD point, where these opposing nonlinearities cancel one another and the fluctuations in the amplitude of vibrations do not locally affect the frequency [5].

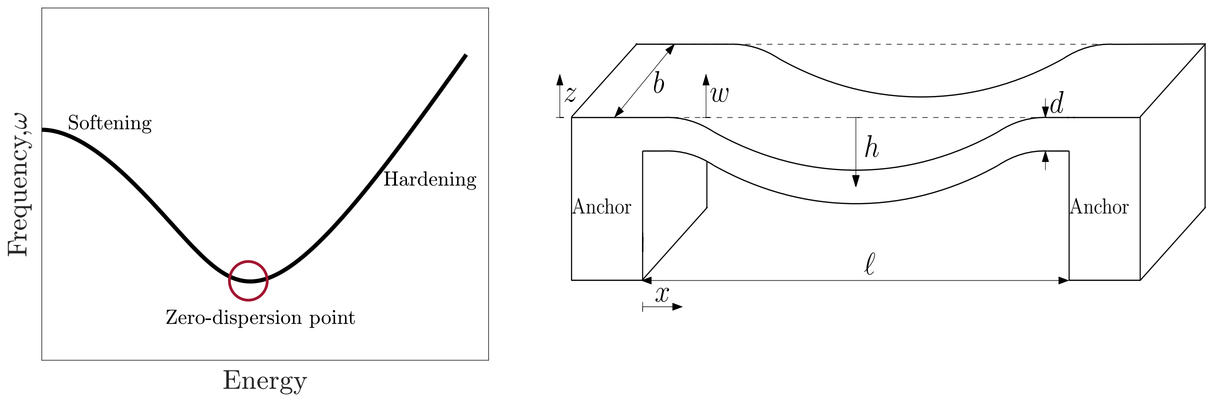


Figure 1: Left panel: The frequency-energy backbone curve of a nonlinear resonator with a zero-dispersion point. At the point of zero-dispersion, $E_{\rm tot} = E_{\rm ZD}$, there is an extremum of the frequency with a zero slope, and hence, the frequency is locally constant. Right panel: Definition sketch of a curved micro-beam. The doubly clamped micro-beam of length $\ell$ and rectangular cross-section $(b \times d)$ have an initial shape of an inverse bell, which is described by a function $w_0(x)$ and a maximal depth of $h = |w_0(\ell/2)|$.

We consider the conservative transverse vibration of a clamped-clamped shallow arch micro-mechanical beam of an initial shape described by $w_0(x)$ with a maximal height of $h$, width $b$, thickness $d$ and length $\ell$ (Fig. 1, right panel). We wish to obtain a reduced-order nonlinear model for the beam and analyze its dynamics using the Euler-Bernoulli beam model. We assume that the flexural motion of the beam is dominated by its fundamental frequency, and use a single-mode approximation, $w(x,t) = q(t)\phi(x)$, where $\phi(x)$ is the eigenfunction of the fundamental mode that satisfies the doubly clamped beam boundary conditions. We perform a Galerkin projection onto $\phi(x)$ to obtain the following nonlinear ordinary differential equation for the modal coordinate $q(t)$

$$\ddot{q} + \omega_0^2 q + \beta q^2 + \gamma q^3 = 0. \tag{1}$$

For an initially bell-shaped beam with $w_0(x) = h\phi(x)$, the coefficients $\omega_0^2$, $\beta$ and $\gamma$ in Eq. (1) are a function of $h$ (the normalized initial elevation of the midpoint of the beam). Moreover, Eq. (1) has an exact analytical solution that describes the strongly nonlinear dynamics of $q(t)$ in terms of elliptic functions [5]. Thus, we can find closed-form expressions for the ZD point $d\omega/dE_{\text{tot}}|_{E_{\text{ZD}}} = 0$ that yield the energy at the ZD point $E_{\text{ZD}}$, and the fundamental frequency at the ZD point $\omega(E_{\text{ZD}})$. We note that both $E_{\text{ZD}}$ and $\omega(E_{\text{ZD}})$ are functions of the coefficients $\omega_0^2$, $\beta$ and $\gamma$, which in turn are functions of $h$. Therefore, using the initial elevation $h$ as the design parameter, we can optimize these two expressions $E_{\text{ZD}} = f(\omega_0^2, \beta, \gamma) = f(h)$ and $\omega_{\text{ZD}} = g(E_{\text{ZD}}) = g(\omega_0^2, \beta, \gamma) = g(h)$. Specifically, by setting $dE_{\text{ZD}}/dh = 0$, we can find the initial elevation $h$ that maximize the energy level of the ZD point, $\max\{E_{\text{ZD}}(h)\}$, and by setting $d\omega_{\text{ZD}}/dh = 0$ we can find the initial elevation $h$ that maximize the frequency of oscillation at the ZD point, $\max\{\omega_{\text{ZD}}(h)\}$. As can be seen form the left panel of Fig. 2, $E_{\text{ZD}}$ is an increasing monotonic function of $h$, and thus, the maximal $E_{\text{ZD}}$ is achieved at the maximal initial elevation that is possible $h_{\max}$. In contrast, $\omega_{\text{ZD}}$ is not a monotonic function of $h$ (Fig. 2, left panel). Therefore, for given dimensions and properties of the beam, there is a unique optimal frequency (Fig. 2, left panel).

Using the bell-shaped beam $w_0(x) = h\phi(x)$ as an initial shape of the curved beam, we apply a genetic algorithm [6] to find the optimal shape of the curved beam that yields a ZD point at the highest frequency and energy level. The genetic algorithm uses the initial shape of the beam to create a population (group of shape functions) of other solutions in its vicinity. After calculating the objective function $d\omega/dE_{\text{tot}}|_{E_{\text{ZD}}} = 0$ for each individual (certain shape function) of the population, the algorithm creates the next generation of population using the fittest solutions of the last generation by: (i) selection, where the fittest solutions survive for the next generation, (ii) crossover, where each two solutions are being used to create a new solution, and (iii) mutation, where some of the solutions are changed randomly. For each shape function, we used spectral methods to calculate the coefficients of Eq. (1) and find its ZD point. After 500 generations of a population of 15 individuals, the algorithm converged to an optimal shape function. The ZD point of the new shape function is achieved at energy levels 3 times higher than the initial bell-shaped beam $w_0(x) = h\phi(x)$ and with a frequency of more than 3 times higher (see Fig. 2, right panel).

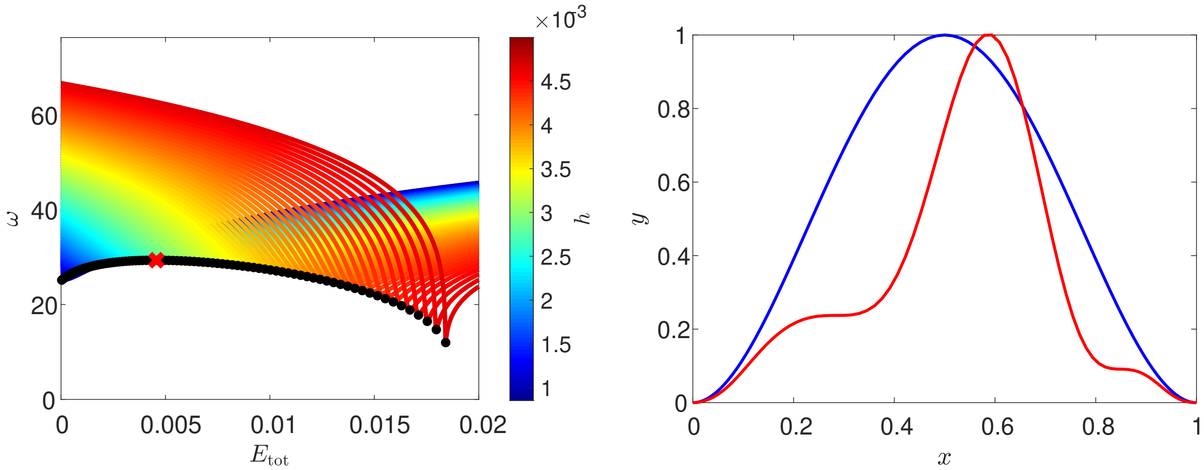


Figure 2: Left panel: The frequency-energy dependency of a bell-shaped beam. The fundamental frequency of oscillation $\omega$ is overlaid by the frequency of oscillation at the ZD point $\omega_{\text{ZD}}$ (black) as a function of the total energy of the system $E_{\text{tot}}$ for different values of initial elevation $h$. The maximal oscillation frequency at the ZD point, $\max\{\omega_{\text{ZD}}(h)\}$, is denoted by the red crossmark on the curve of $\omega_{\text{ZD}}$. Right panel: Comparison between the initial bell-shaped function of the beam (in blue) and the optimal shape function after 500 generations of a population of 15 individuals (in red). The frequency and energy of the zero-dispersion point in the optimal shape are threefold higher than in the bell-shaped beam.

## References


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