Nonsmooth Modal Analysis of a Rectangular Plate in Unilateral Contact

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Summary. A scheme for the nonsmooth modal analysis of a rectangular plate in frictionless unilateral contact with a rigid foundation is demonstrated. Application of nonsmooth modal analysis requires finding periodic solutions to the Signorini problem. To this end, the nodal boundary method in the framework of the finite element method is used. It allows to approximate the solution through a system of nonsmooth ordinary differential equations in the internal displacements of the plate. The resulting system exhibits chattering-free periodic solutions numerically found via the harmonic balance method. Sequential continuation is used for detection of nonsmooth modes.

Introduction

Nonsmooth modal analysis (NMA) refers to the application of nonlinear modal analysis on structures for which the dynamics is governed by nonsmooth equations [7]. In the present work, the subject of analysis is a rectangular plate, with in-plane displacements, in unilateral contact with a rigid obstacle expressed via the usual Signorini complementarity conditions [1]. In essence, NMA requires the detection of families of periodic motions to the autonomous system [5, 7, 9]. Generally, NMA requires use of numerical techniques since closed-form solutions to the Signorini problem do not generally exist [7, 9]. In [9], the wave finite element method (WFEM) was used for NMA of the bar and has proven to provide accurate result for the case of the uniform-area bar. However, extension of this methodology to two-dimensional systems has failed due to high numerical energy dissipation effectively annihilating periodic solutions [9]. The finite element method (FEM) has also been commonly used to solve the Signorini problem numerically [3]. In the FEM framework, there exist several numerical schemes for treatment of the Signorini conditions. However, those were not used for NMA for various reasons. For example, schemes utilizing a fully elastic Newtonian impact law exhibit non-physical chattering [3, 7] while the penalty [3] and Nitsche [2] methods do not exactly enforce impenetrability with the obstacle in the discrete setting. The mass redistribution method (MRM) [4], while allowing for energy conservation and elimination of chattering, requires solving a constrained optimization problem in many variables to form the reduced mass matrix. However, the *Nodal Boundary Method* (NBM) eliminates chattering by reducing the mass and stiffness matrices via a series of linear operations on the mass matrix's rows and columns [8]. The ordinary differential equations (ODEs) resulting from NBM exhibit periodic solutions which can be found via the harmonic balance method (HBM) [8]. Continuous families of periodic solutions are then found via sequential continuation along different periods of the motion [5].

Boundary Value Problem for Periodic Motions with Unilateral Contact

Within the framework of two-dimensional in-plane elasticity, we investigate the problem of a thin and isotropic plate prone to unilateral contact with a rigid foundation. The equation governing the motion of the plate reads

$$\rho\bar{\mathbf{u}}_{tt}(\mathbf{x},t) - \mathbf{div}(\boldsymbol{\sigma}(\bar{\mathbf{u}}(\mathbf{x},t))) = \mathbf{0}, \quad (\mathbf{x},t) \in \Omega \times [0,\infty) \tag{1}$$

where $\bar{\mathbf{u}}(\mathbf{x},t) : \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{R}$ and ρ describe the displacement field and density, respectively. The stress tensor $\boldsymbol{\sigma}(\bar{\mathbf{u}}(\mathbf{x},t))$ is related to the displacements via a classical plane-stress assumption. Other than the classic Neumann and Dirichlet boundary conditions, a portion Γ_C of the plate's boundary is prone to the unilateral contact conditions

$$0 \leq g_0 - \bar{\mathbf{u}}(\mathbf{x},t)\cdot\mathbf{n}, \quad \sigma_n \leq 0, \quad (g_0 - \bar{\mathbf{u}}(\mathbf{x},t)\cdot\mathbf{n})\sigma_n(\bar{\mathbf{u}}(\mathbf{x},t)) = 0, \quad \mathbf{x} \in \Gamma_\mathrm{C} \tag{2}$$

where g_0 defines the distance between the non-deformed boundary and the rigid obstacle. The contact pressure is captured by $\sigma_n(\bar{\mathbf{u}}(\mathbf{x},t)) = \mathbf{n}^\top\boldsymbol{\sigma}(\bar{\mathbf{u}}(\mathbf{x},t))\mathbf{n}$ where $\mathbf{n}$ is the outward normal to the contact boundary. For NMA, we search for solutions of period T such that $\bar{\mathbf{u}}(\mathbf{x},0) = \bar{\mathbf{u}}(\mathbf{x},T)$ and $\bar{\mathbf{u}}_t(\mathbf{x},0) = \bar{\mathbf{u}}_t(\mathbf{x},T)$.

Nodal Boundary Method

The NBM applies on the FEM formulation of the Signorini problem, stated in Equations (1) and (2), which takes the form

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{G}^\top\boldsymbol{\lambda}(t) \tag{3}$$

$$\boldsymbol{\lambda}(t) + \mathbf{max}(\mathbf{0}, g\mathbf{1} - \mathbf{u}_\mathrm{C}(t) - \boldsymbol{\lambda}(t)) = \mathbf{0} \tag{4}$$

where $\mathbf{u}(t) \in \mathbb{R}^N$ constitute the N nodal displacements approximating the displacement field $\bar{\mathbf{u}}(\mathbf{x},t) \approx \mathbf{P}(\mathbf{x})\mathbf{u}(t)$ with $\mathbf{P}(\mathbf{x})$ denoting piecewise-Lagrangian shape functions. In the NBM, we distinguish between nodal displacements $\mathbf{u}_\mathrm{C}(t)$ on Γ_C and the remainder of the nodal displacements $\mathbf{u}_\mathrm{O}(t)$ such that $\mathbf{u}(t) = (\mathbf{u}_\mathrm{O}(t)\ \mathbf{u}_\mathrm{C}(t))^\top$. It is well known that formulation (3) and (4) form an ill-posed problem due to the existence of infinitely many values for $\boldsymbol{\lambda}$ at the moment of contact [1]. In the NBM, we attempt to resolve this ill-posedness by assigning $\boldsymbol{\lambda}$ a relationship with $\mathbf{u}(t)$ via the FEM approximation of the boundary integral on Γ_{C}:

$$\mathbf{G}^{\mathrm{T}}\boldsymbol{\lambda} \leftarrow \int_{\Gamma_{\mathrm{C}}} \mathbf{P}^{\top}(\mathbf{x})\sigma_n(\mathbf{P}(\mathbf{x})\mathbf{u}(t))\mathrm{d}\mathbf{x} = \mathbf{G}^{\top}(\mathbf{N}_{\mathrm{C}}\mathbf{u}_{\mathrm{C}}(t) + \mathbf{N}_{\mathrm{O}}\mathbf{u}_{\mathrm{O}}(t)) \tag{5}$$

where $\mathbf{N}_{\mathrm{C}}$ and $\mathbf{N}_{\mathrm{O}}$ are constant matrices. Substitution of (5) into (4) results in an LCP in $\mathbf{u}_{\mathrm{C}}(t)$ admitting a unique solution for given $\mathbf{u}_{\mathrm{O}}(t)$ and g. This unique solution consists of piecewise constant matrix $\mathbf{A}(\mathbf{u}_{\mathrm{O}}(t), g)$ and vector $\mathbf{d}(\mathbf{u}_{\mathrm{O}}(t), g)$ such that $\mathbf{u}_{\mathrm{C}}(t) = \mathbf{A}(\mathbf{u}_{\mathrm{O}}(t), g)\mathbf{u}_{\mathrm{O}}(t) + g\mathbf{d}(\mathbf{u}_{\mathrm{O}}(t), g)$. The values of $\mathbf{A}$ and $\mathbf{d}$ can be found numerically by solving the LCP. From this expression, we construct the NBM approximation of the displacement which always satisfies (4) and (5) along with the FEM approximation of the Signorini conditions

$$\bar{\mathbf{u}}(\mathbf{x}, t) \approx \mathbf{P}(\mathbf{x})(\mathbf{A}^*(\mathbf{u}_{\mathrm{O}}, g)\mathbf{u}_{\mathrm{O}}(t) + \mathbf{d}^*(\mathbf{u}_{\mathrm{O}}, g)), \quad \mathbf{A}^*(\mathbf{u}_{\mathrm{O}}, g) = \begin{bmatrix} \mathbf{I} \\ \mathbf{A}(\mathbf{u}_{\mathrm{O}}, g) \end{bmatrix}, \quad \mathbf{d}^*(\mathbf{u}_{\mathrm{O}}, g) = \begin{bmatrix} \mathbf{0} \\ \mathbf{d}(\mathbf{u}_{\mathrm{O}}, g) \end{bmatrix}. \tag{6}$$

In the FEM framework, substitution of approximation (6) into (3) yields the system of ODEs

$$(\mathbf{A}^*(\mathbf{u}_{\mathrm{O}}, g))^{\top}(\mathbf{M}\mathbf{A}^*(\mathbf{u}_{\mathrm{O}}, g)\ddot{\mathbf{u}}_{\mathrm{O}}(t) + \mathbf{K}\mathbf{A}^*(\mathbf{u}_{\mathrm{O}}, g)\mathbf{u}_{\mathrm{O}}(t) + \mathbf{K}\mathbf{d}^*(\mathbf{u}_{\mathrm{O}}, g)) = \mathbf{0} \tag{7}$$

which are nonsmooth ODEs (due to $\mathbf{A}^*$ and $\mathbf{d}^*$) in $\mathbf{u}_{\mathrm{O}}(t)$. Periodic solutions of the ODE can be found via the HBM [8]. Then, sequential continuation is used to determine families of periodic solutions of the plate in unilateral contact, in a similar fashion to [5].

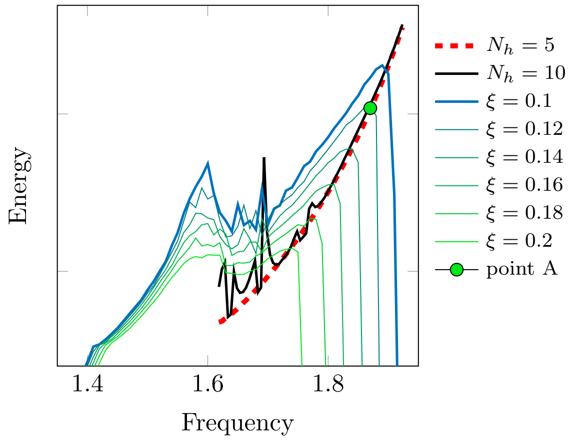

Figure 1: Forced response of the plate for various damping coefficients ξ and backbone curves generated via NBM-HBM with varying number of harmonics N_h.

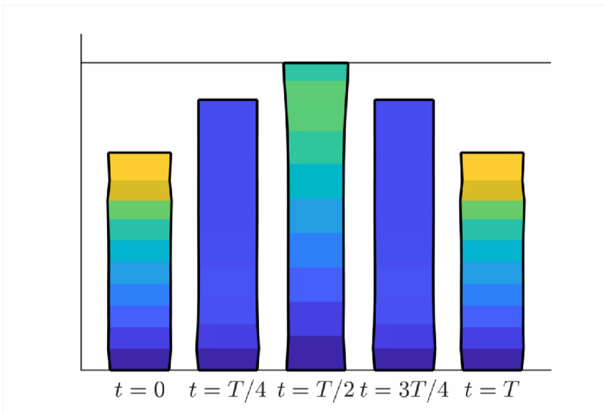

Figure 2: Solution at point A in Figure 1 at different times fractions of the period $T = 2\pi/1.87$. Color gradient represents $||\mathbf{u}(\mathbf{x}, t)||_2$

Conclusions

Under the NBM's formulation of the Signorini problem, NMA of the plate was conducted successfully. Indeed, this formulation may apply to more intricate structures prone to unilateral contact. Therefore, future work shall include expanding the NBM formulation to practical engineering cases. Also, comparison of NBM against Nitsche method for purposes of NMA is considered. While Nitsche applies the Signorini conditions in a weak sense, the NBM does so in a strong sense and it is of interest to compare the rates of convergence of both methods to the true solution.

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