Stabilizability Limits for the Inverted Pendulum with a Multiple-Delay Fractional-Order Controller

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Summary. In this study, we consider the PD$^\mu$ control of the inverted pendulum with different delays in the proportional and the fractional derivative terms. This concept gives a transition between several special cases already investigated in the literature. The main question is whether the critical delay can further be extended by employing fractional-order feedback combined with delay detuning.

Introduction

Time delay in state-feedback systems sets a strong limitation in the stabilization of unstable plants. If the feedback delay is larger than some critical value, then the system cannot be stabilized. This feature can well be demonstrated by the inverted pendulum paradigm [3, 4]. Stabilization of an inverted pendulum by proportional-derivative (PD) feedback is possible if and only if the feedback delay τ is smaller than a critical delay given by

$$\tau_{\rm crit}^{\rm PD} = \frac{T}{\pi\sqrt{2}}, \tag{1}$$

where T is the period of the oscillations of the pendulum hung downwards [4]. If $\tau > \tau_{\rm crit}$, then one cannot find proportional and derivative gains that stabilizes the inverted position of the pendulum.

The critical delay can be increased by employing control laws other than PD feedback. For instance, if the feedback involves acceleration (PDA feedback), then the critical delay can be increased to $\tau_{\rm crit}^{\rm PDA} = \sqrt{2}\,\tau_{\rm crit,PD}$ [3]. Alternatively, if the delay of the proportional and the derivative terms are detuned, then the critical delay increases to $\tau_{\rm crit}^{\rm dPD} \approx 1.47\,\tau_{\rm crit}^{\rm PD}$ [3]. Another alternative way to increase the critical delay is the application of fractional-order control: in case of PD$^\mu$ feedback, $\tau_{\rm crit}^{\rm PD^\mu} \approx 1.12\,\tau_{\rm crit}^{\rm PD}$ [1].

Introducing fractional-order derivative in the feedback loop allows us to exploit the time history starting from some initial time to the current time instant. This can be seen from the most frequently used definitions of fractional derivative: the Riemann-Liouville fractional derivative, the Caputo fractional derivative and the Grünvald-Letnikov fractional derivative. All of these definitions of the fractional derivative resembles a distributed delay term that converts into a point delay term if the order of the derivative is an integer [2].

Problem statement

The characteristic function of the system under investigation reads

$$D(s) = s^2 - a_0 + k_{\rm p}{\rm e}^{-s\tau_{\rm p}} + k_{\rm d}s^\mu{\rm e}^{-s\tau_{\rm d}}\,, \tag{2}$$

where $a_0 > 0$ is the open-loop system parameter, $\tau_{\rm p} > 0$ and $\tau_{\rm d} > 0$ are the feedback delays and $0 < \mu < 2$ is the order of the fractional derivative. This system can also be interpreted as a control system with a single latency τ with some additional delays (delay detunings) $\delta_{\rm p} \geq 0$ and $\delta_{\rm d} \geq 0$ in both terms such that $\tau_{\rm p} = \tau + \delta_{\rm p}$, $\tau_{\rm d} = \tau + \delta_{\rm d}$.

The D-subdivision method can also be applied to fractional-order systems. Substitution of $s = 0$ and $s = \pm{\rm i}\omega$, $\omega > 0$ into $D(s) = 0$ gives the D-curves

$$s = 0\ :\quad k_{\rm p} = a_0\,,\quad k_{\rm d} \in \mathbb{R}\,, \tag{3}$$

$$s = \pm{\rm i}\omega, \omega > 0\ :\quad \begin{cases} k_{\rm p} = \left(a_0 + \omega^2\right)\dfrac{\sin(\frac{\mu\pi}{2} - \tau_{\rm d}\omega)}{\sin\left(\frac{\mu\pi}{2} - (\tau_{\rm d} - \tau_{\rm p})\omega\right)}\,, \\ k_{\rm d} = \left(a_0 + \omega^2\right)\dfrac{\sin(\tau_{\rm p}\omega)}{\omega^\mu\sin\left(\frac{\mu\pi}{2} - (\tau_{\rm d} - \tau_{\rm p})\omega\right)}\,. \end{cases} \tag{4}$$

The D-curves bounds the parameter regions in the plane $(k_{\rm p}, k_{\rm d})$ where the number of unstable characteristic roots is constant. Stable regions (zero unstable characteristic roots) can be determined numerically using the argument principle. When the delays increase then the stable regions typically shrink and disappear. There is a critical delay $\tau_{\rm crit}^{\rm dPD^\mu}$: if $\min(\tau_{\rm p}, \tau_{\rm d}) > \tau_{\rm crit}^{\rm dPD^\mu}$ then the system cannot be stabilized by any triplet $(k_{\rm p}, k_{\rm d}, \mu)$. The goal of this study is to determine the stabilizability boundaries in the plane $(\tau_{\rm p}, \tau_{\rm d})$ and to find $\tau_{\rm crit}^{\rm dPD^\mu}$.

Special case: PD$^\mu$ controller with a single delay

Stabilizability was already investigated if the delays in the proportional and fractional derivative terms are the same ($\tau_{\rm p} = \tau_{\rm d} = \tau$). In the case of a PD$^\mu$ controller with a single delay, the stabilizable region was derived in [1] in the plane of the dimensionless parameters $a = a_0\tau^2$ and μ (see the left panel in Figure 1).

Using the D-subdivision technique, we can observe four types of loss of stabilizability. These geometric conditions can be directly translated into the multiplicity conditions shown in the right panel of Figure 1. This gives a more uniform description of the stabilizability boundaries compared to that of [1]. Conditions $\det \mathbf{J} = 0$ corresponds to the singularity of the Jacobian matrix of the other three (four) equations with respect to $k_{\rm p}, k_{\rm d}, \omega_1$ (and ω_2). The geometric interpretation of this condition is the tangency of D-curves at the limit of stabilizability.

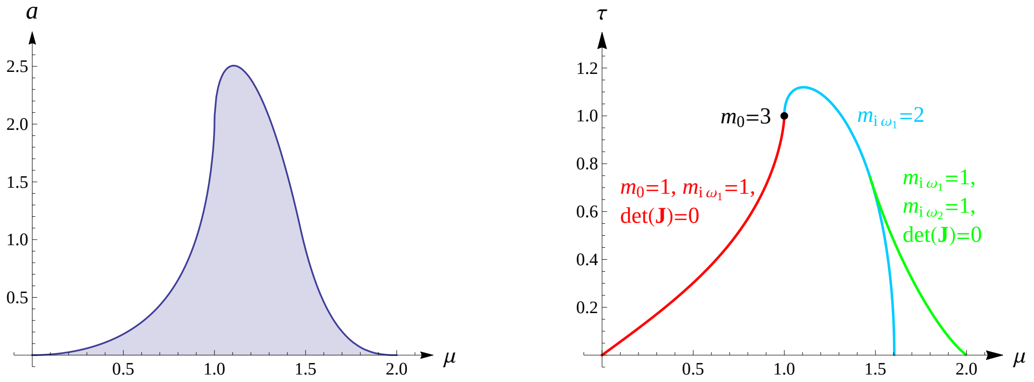

Figure 1: Stabilizable region of (2) if $\tau_{\rm p} = \tau_{\rm d} = \tau$ with $a = a_0\tau^2$ (left). The stabilizability boundaries and multiplicity conditions in the plane (μ, τ) if $a_0 = 2$ (right).

Main results: stabilizability diagrams in the plane $(\tau_{\rm p}, \tau_{\rm d})$

Using a similar technique described in the previous section, we can construct stabilizability diagrams for the case $\tau_{\rm p} \neq \tau_{\rm d}$. First, we need to detect the geometric conditions at the limit of stabilizability using D-subdivision. These geometric conditions can be translated into multiplicity conditions. From the multiplicity conditions, we obtain a nonlinear system of equations, which can be reduced after solving for $k_{\rm p}$ and $k_{\rm d}$. Finally, the reduced equations can be solved using pseudo-arclength continuation.

Figure 2 shows the stabilizability boundaries in the plane $(\tau_{\rm p}, \tau_{\rm d})$ for different values of μ. The stabilizable region can be extended compared to the detuned PD controller ($\mu = 1$) by choosing an appropriate value of the fractional order μ. The largest admissible delay is obtained for $\mu = 0.999637$. In this case the critical delay is $\tau_{\rm crit}^{{\rm dPD}^\mu} = 1.00778\tau_{\rm crit}^{\rm dPD}$ (see the right panel of Figure 2). Hence, an extremely small but still finite extension of the critical delay can be achieved by employing detuned fractional-order control.

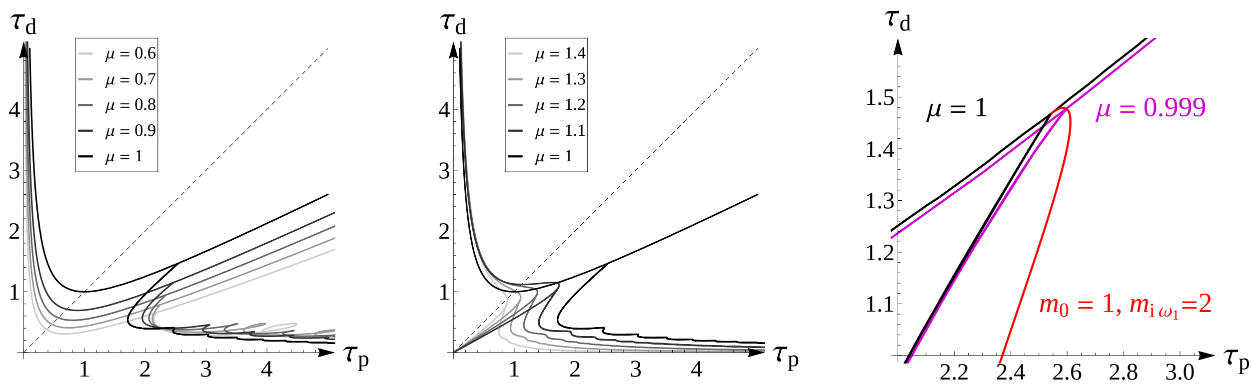

Figure 2: The stabilizability boundaries in the plane $(\tau_{\rm p}, \tau_{\rm d})$ if $\mu \leq 1$ (left) and $\mu \geq 1$ (middle) with $a_0 = 2$ (stabilizable regions are to the left of the curves). The path of the critical point associated with the maximal allowed delay in the plane $(\tau_{\rm p}, \tau_{\rm d})$ if $\mu \leq 1$ (right).

References

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