Tracking basin boundaries with Clustered Simple Cell Mapping method

Gergely Gyebrószki† and Gábor Csernák†
†*MTA-BME Research Group on Dynamics of Machines and Vehicles,*
H-1111, Budapest, Műegyetem rkp 3.

Summary. The Simple Cell Mapping (SCM) method [3] is a simple and effective algorithm to analyse the (discretized) state space of dynamical systems. A state space region is divided into cells and an image cell (where the dynamics lead to) is determined for each cell. SCM is able to find chaotic attractors and – depending on the cell resolution – repellers, and their basin of attraction. Chaotic structures are usually represented by a periodic cell group with high period, while basins of attraction are composed of transient cell sequences leading to periodic groups.
Previously, we have improved the SCM method with the ability to extend its underlying state space region and join the new state space region to the previous SCM solution. This method is called Clustered Simple Cell Mapping (CSCM) [2]. We show that CSCM can be a valuable tool for automatically exploring the state space or tracking certain features – for example the basin of attraction of an attractor or the closure of a chaotic repeller, especially in the case, where the underlying system can exhibit crisis bifurcations. An example is shown using the well-known Ikeda-map [4].

Clustered Simple Cell Mapping

Simple Cell Mapping (SCM) is a great tool to quickly analyse the state space of dynamical systems. An initial state space region is discretized into cells, for each cell a single image cell is determined, usually by following a trajectory from the center point of each cell for a given time step, or by applying the map corresponding to discrete systems. The SCM method then classifies cells as either periodic cells (belonging to a periodic group) or transient cells (leading to one of the periodic groups within the state space, or the region outside – the sink cell). Chaotic attractors or repellers are usually represented by a periodic cell group with high periodicity.
There are several extensions for the SCM method, some of them exploits certain properties of a class of dynamical systems: eg. discontinuous systems [5], or Filippov systems [1]. In [2] we have introduced the Clustered Simple Cell Mapping (CSCM) method, another extension aiming to adaptively extend the analysed state space region in a computationally effective manner. Clustered SCM is able to join an additional state space region to the so called *cluster* of SCMs, and update the solution by re-using the existing cell classification in the initial SCM region.
The concept is illustrated in Fig. 1: the left region is the initial SCM solution containing a periodic group (denoted by dark grey ■) and a set of transient cells leading to it (gray colour ■). The right state space region is added to the cluster of SCMs: this allows the discovery of transient cells leading to a previously classified periodic group (green ■), a new periodic group (at the boundary of the two regions) and non-trivial transients in the initial SCM's region (denoted by orange ■).
Clustered Simple Cell Mapping allows the continuation of SCM solutions towards state space structures (e.g. basins of attraction) that get clipped by the initial choice of state space region.

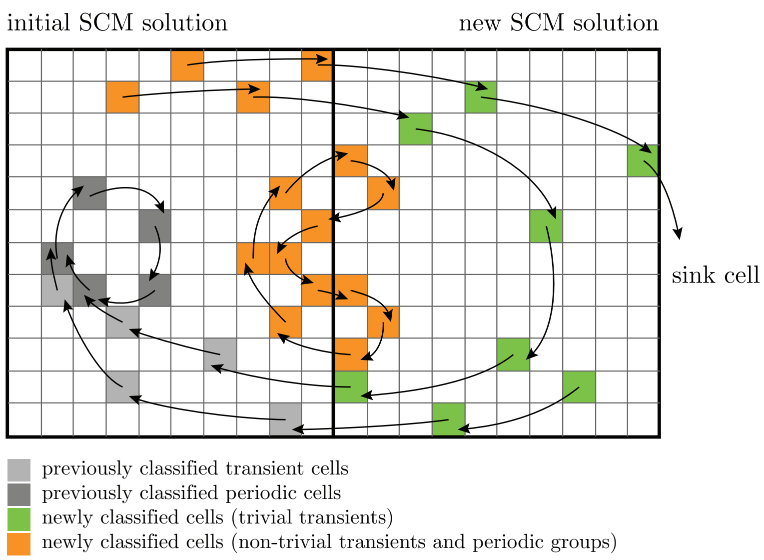

Figure 1: The concept of the Clustered Simple Cell Mapping (CSCM) method.

The Ikeda-map

The Ikeda-map [4] is originated from a model of light going around across a nonlinear optical resonator. Depending on its parameters it can have fixed points, chaotic attractors or a repeller in its state space and can exhibit crisis bifurcations.

$$x_{n+1} = 1 + u\,(x_n\,\cos(t_n) - y_n\,\sin(t_n)) \qquad (1)$$
$$y_{n+1} = u\,(x_n\,\sin(t_n) + y_n\,\cos(t_n)),$$

where u is a parameter and:

$$t_n = 0.4 - \frac{6}{1 + x_n^2 + y_n^2}. \qquad (2)$$

An application example

We show an example of state-space exploration by using the CSCM method in the case of the Ikeda-map defined by Eq. (1). The starting region (denoted by 1 in Fig. 2) is a 6×6 square at $(0, 0)$ in the (x, y) plane, where a chaotic repeller resides. Note, that the repeller can only be found by SCM if the cell resolution is sufficiently coarse, to artificially *stabilize* it. The adaptive state-space extension algorithm adds another 8 square-shaped state space region to the cluster, by following the basin of attraction along the four main directions. This is shown on Fig. 2, where the successive state-space regions are numbered. After the initial exploration procedure, the cluster is made convex by adding regions 10 to 12. As the last state space region is included – where a fixed point resides – the SCM algorithm is able to classify its basin of attraction as well (denoted by shades from green to blue based on step numbers needed to reach the fixed point).

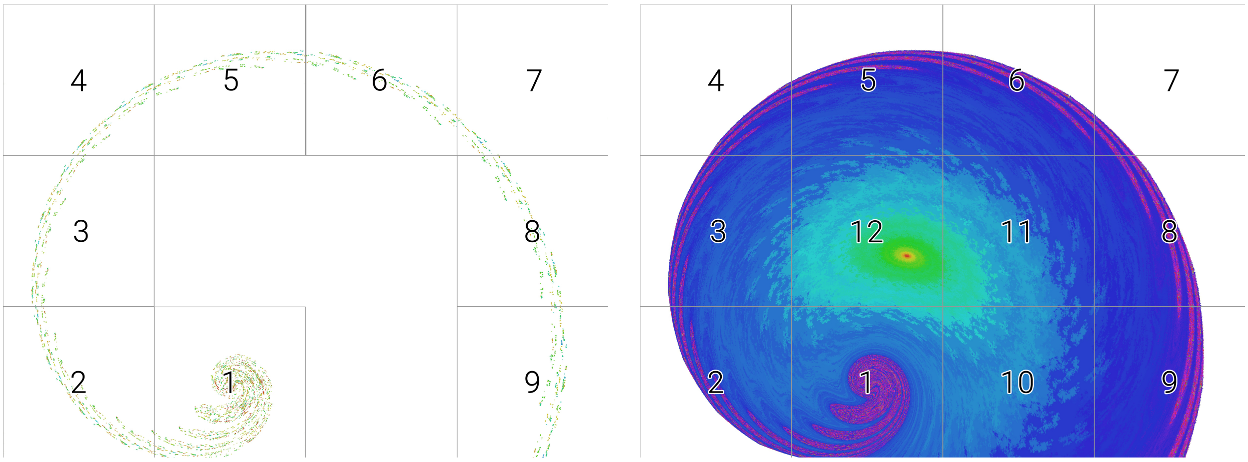

Figure 2: A CSCM cluster consisting of 9 SCM solutions (*left*) and 12 SCM solutions (*right*) of the Ikeda-map with $u = 0.96$. The state space region consists of squares with side lengths of 6.0. The first region's center is at $(0, 0)$. The whole cluster spans from $(x_0, y_0) = (-9, -3)$ to $(x_1, y_1) = (15, 15)$.

Conclusions

Clustered Simple Cell Mapping is a method allowing the successive expansion of the state space region corresponding to an SCM solution of a dynamical system. This allows quick and computationally effective exploration of the state space in case of crisis bifurcations (where certain state space features *explode*), or tracking basins of attraction as shown in Fig. 2. Since existing SCM solutions are re-used when the cluster is expanded, CSCM allows interactive or real-time applications.

Acknowledgements

The research reported in this paper and carried out at BME has been supported by the Hungarian National Research, Development and Innovation (NRDI) Office under Grant No. NKFI-128422.

References

[1] Erazo, C., Homer, M.E., Piiroinen, P.T., Di Bernardo, M. (2017). Dynamic cell mapping algorithm for computing basins of attraction in planar filippov systems. International Journal of Bifurcation and Chaos **27**(12), 1730041

[2] Gyebrószki, G., Csernák, G. (2017). Clustered simple cell mapping: An extension to the simple cell mapping method. Communications in Nonlinear Science and Numerical Simulation **42**, 607–622

[3] Hsu, C.: Cell-to-Cell Mapping: A Method of Global Analysis for Nonlinear Systems, *Applied Mathematical Sciences*, vol. 64. Springer, Singapore (1987)

[4] Ikeda, K., Daido, H., Akimoto, O. (1980). Optical turbulence: Chaotic behavior of transmitted light from a ring cavity. Phys. Rev. Lett. **45**, 709–712

[5] de Kraker, B., van der Spek, J.A.W., van Campen, D.H.: Extensions of cell mapping for discontinuous systems. In: M. Wiercigroch, B. de Kraker (eds.) Applied Nonlinear Dynamics and Chaos of Mechanical Systems with Discontinuities, chap. 4, pp. 61–102. World Scientific (2000)