Small and large amplitude, free oscillations of a pinned spherical interface

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Summary. We investigate using linearised theory and numerical solution to the incompressible Euler equation, the effect of density ratio and perturbation amplitude on shape oscillations of a pinned spherical interface. The interface in the base state is a spherical cap separating quiescent fluids and may be pinned at any angle lying in the interval $[0, \pi]$. The linearised analysis presented in [1] which takes into account the density of fluid inside the spherical cap only, is further extended to account for the density of fluids inside($\rho^{\mathcal{I}}$) as well as outside($\rho^{\mathcal{O}}$). The limits of linearised theoretical predictions are tested via numerical simulations by exciting the first eigenmode of the system at various pinning angles(α). For sufficiently small perturbation amplitude, the agreement with linearised predictions found to be quite good. A systematic nonlinear correction to frequency is observed as the perturbation amplitude is increased, becoming particularly discernable after the first few oscillations. The simulations are carried out using the open source code Gerris [8] and we quantify the effect of non-linearity and droplet ejection is observed due to increasing perturbation amplitude. We investigate the bubble limit and the droplet limit respectively where density of fluid inside the cap is negligible compared to that outside and vice versa.

Inviscid oscillations of a pinned interface

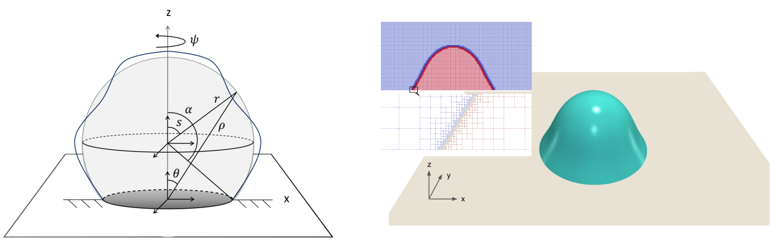
Natural and forced oscillations of a pinned spherical cap due to surface tension is of interest in numerous applications both in engineering as well as in the natural sciences and remains a topic of active research interest [2]. The frequency spectrum of small amplitude free oscillations of an inviscid free drop of unperturbed radius R_0 and density $\rho^{\mathcal{I}}$ due to the restoring force of surface tension T was first obtained by Rayleigh [9]. The effect of the second fluid on the spectrum was studied by Lamb [5] lead to the classical dispersion relation for shape oscillations of liquid drops ($\rho^{\mathcal{I}} >> \rho^{\mathcal{O}}$) and bubbles ($\rho^{\mathcal{O}} >> \rho^{\mathcal{I}}$):

$$\omega_0^2 = \frac{T}{R_0^3}\frac{l(l+1)(l-1)(l+2)}{\rho^{\mathcal{I}}(l+1) + l\rho^{\mathcal{O}}} \tag{1}$$

where l is a positive integer arising from the spherical harmonic $\mathbb{Y}_l^m(s,\psi)$. In many engineering applications like vibration induced atomisation [4], it is necessary to know the modification to the Rayleigh-Lamb frequency spectrum (eqn.1) due to the droplet resting on a substrate. While a number of studies have investigated this problem under the assumption of a hemispherical cap [6], the first study which systematically studied the modification to the spectrum for the case of a droplet pinned at an arbitrary angle α was by Steen [1]. As a part of the present study, we extend the analysis of of their work to take into account the density of the second fluid on the shape oscillation of pinned bubbles. For this, we closely follow the analysis by Steen employing two spherical coordinate systems as shown in fig.1(a). It maybe shown that in the inviscid, irrotational approximation [3, 7] the perturbation velocity potential ϕ and the surface perturbation η can be expressed in the standing wave form [3].

$$\eta(s,\psi,t) = a(t)y(s)\exp(il\psi),\ \ \phi^{\mathcal{I}}(\rho,\theta,\psi,t) = \dot{a}(t)\ \Phi^{\mathcal{I}}(\rho,\theta)\exp(il\psi),\quad \phi^{\mathcal{O}}(\rho,\theta,\psi,t) = \dot{a}(t)\ \Phi^{\mathcal{O}}(\rho,\theta)\exp(il\psi) \tag{2a,b,c}$$

Here, $\dot{a} \equiv \frac{da}{dt}$ and it maybe be further shown that $a(t)$ satisfies the simple harmonic oscillator equation $\ddot{a}+\omega_0^2 a(t) = 0$. The pinned frequency ω_0^2 is calculated by solving a generalised eigenvalue problem employing the Green function formalism suggested in [1] and additionally also taking into account both densities [3]. We test predictions made in the droplet and in the bubble limits using the open-source code Gerris [8] and these are described below in fig. 2, 3 and 4.



(a) Geometry (b) Initial condition in numerical simulation

Figure 1: (a) A cartoon representation of a spherical cap (in the base state) pinned at an angle α on a substrate [1]. In close analogy to the linearised analysis by [1], we employ two spherical coordinates systems viz. one centred on the substrate with scaled coordinates (ρ,θ,ψ) and another centred at the spherical cap with scaled coordinates (r,s,ψ). The density of the fluid inside is taken as $\rho^{\mathcal{I}}$ while that outside is $\rho^{\mathcal{O}}$. (b) The initial deformation is obtained from linear theory by computing the eigenmodes which may be represented as $r = R_0 + a_0\eta(s,\psi)$. Shown here is the lowest axisymmetric eigenmode for a pinning angle of 80°. For consistency and to justify the neglect of gravity, R_0 is chosen to be much smaller than the capillary length scale. Grid density of 2048x2048 is used near the interface and 512x512 is used away from the interface.

Droplet Limit $(\rho^{\mathcal{O}}/\rho^{\mathcal{I}} \to 0)$**: Axisymmetric modes and comparison of linear theory with simulations**

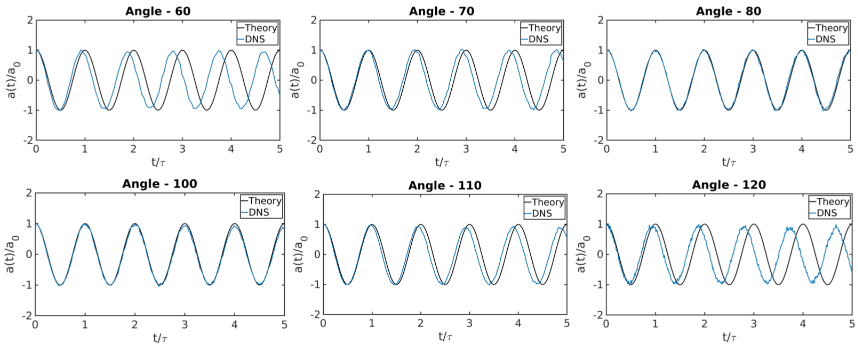


Figure 2: Time signal from oscillatory response of the interface obtained from numerical solution to the incompressible Euler equation (labelled as DNS in the plots) with surface tension ($72dyne/cm$). We distort the spherical cap at the indicated pinning angle using the axi-symmetric eigenmodes with zero perturbation velocity as shown in fig.1b. In the droplet case, these eigenmodes are provided in [1] and correspond to the droplet limit. ($R_0 = 0.1cm, \rho^{\mathcal{O}}/\rho^{\mathcal{I}} = 0.001$. The time signals are measured at the interface at a location slightly off the apex of the spherical cap and compared to a sinusoidal signal with the frequency obtained by solving a generalised eigenvalue problem in Matlab [1, 3]. Note that the time t is non-dimensionalised by the time period τ predicted from linear theory [1]. A good agreement with linear predictions is seen in simulations with small disagreement in the predicted frequency around $\alpha = 60°$ and $\alpha = 120°$.

Bubble Limit $(\rho^{\mathcal{O}}/\rho^{\mathcal{I}} \to \infty)$**: axisymmetric modes and comparison with simulations**

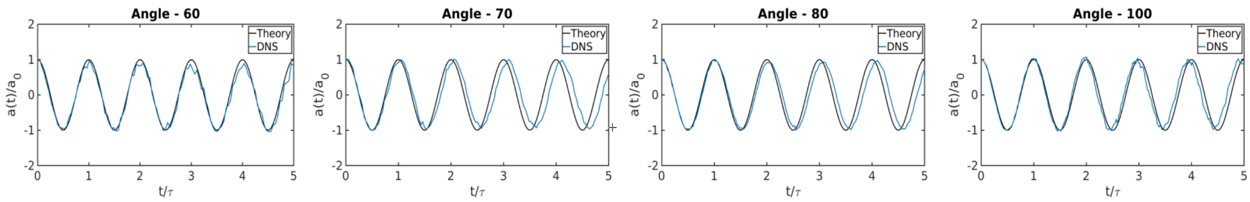
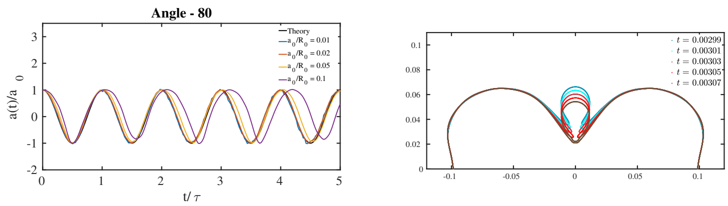


Figure 3: Oscillatory response obtained from numerical simulations in the bubble limit($\rho^{\mathcal{O}}/\rho^{\mathcal{I}} = 1000$) and comparison with linearised predictions [3]. A resonably agreement is seen in all cases for comparison of linear theory with simulations.



(a) Effect of changing perturbation amplitude (b) Droplet ejection during oscillation for $a_0/R_0 = 0.3$

Figure 4: Effect of change of perturbation amplitude for a spherical cap pinned at $80°$.

Fig. 4(a) shows the effect of increase of the perturbation amplitude a_0. It is seen that the time signal becomes distinctly non-sinusoidal with a frequency which increases with increasing a_0. Panel (b) of the same figure shows a large amplitude simulation where a droplet is (nearly) detached from the spherical cap as a result of large amplitude deformation. Systematic comparisons of nonlinear effects for free and pinned droplets will be discussed at the meeting.

Conclusions

We extend the theoretical formalism developed in [1] to include the effect of internal as well as external (linearised) fluid inertia for pinned spherical interfaces. For sufficiently small amplitude, we find good agreement with linearised theoretical predictions both in the droplet as well as the bubble limit. For perturbation amplitude exceeding about 10 percent of the spherical cap radius, discernable nonlinear effects are seen in the numerical simulations and in the large amplitude regime, droplet ejection is observed.

References

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