Compact weighted residual formulation for periodic solutions of systems undergoing unilateral contact and frictional occurrences

Mathias Legrand[1] and Christophe Pierre[2]
[1] *Department of Mechanical Engineering, McGill University, Montreal, Canada*
[2] *Department of Mechanical Engineering, Stevens Institute of Technology, Hoboken, New Jersey, USA*

Summary. A very compact weighted residual formulation is proposed for the construction of periodic solutions of oscillators subject to unilateral contact and frictional occurrences. The key idea is to express all governing equations as equalities, which can then be satisfied in a weighted residual sense.

Toy system

In order to introduce the proposed formulation, a simple toy system, illustrated in Figure 1, is considered. It is a simple geometrically nonlinear mass-spring system lying on a moving belt of constant linear velocity v. The mass is denoted by m and the stiffness, by k. The length of the spring at rest (in a vertical position when the mass is lying on the belt) is L. The vertical displacement of the mass is $y(t)$ and vertical separation from the belt is possible. The horizontal displacement is $x(t)$. The position at rest is $(x, y) = (0, 0)$. The frictional force acting on the mass is $r_\mathrm{T}(t)$ while the unilateral contact

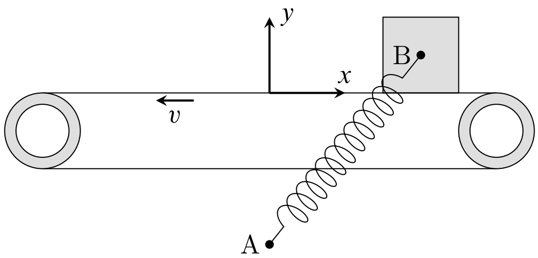

Figure 1: One-dimensional mass-spring moving on a rotating belt. Point A is fixed. The position of point B is $(x(t), y(t))$. The belt is assumed to be infinitely long.

force is $r_\mathrm{N}(t)$. The relative tangential velocity between the mass and the belt is $\dot{u}_\mathrm{T}(t) = -v + \dot{x}(t)$. In the remainder, time is often dropped for readability purposes.

Signorini unilateral contact and Coulomb's friction conditions expressed as equalities
Given the parametrization of the system of interest, the classical Signorini conditions are expressed as the complementarity conditions

$$r_\mathrm{N} \geq 0, \quad y \geq 0, \quad r_\mathrm{N} \cdot y = 0 \tag{1}$$

which can be recast into various equivalent nonsmooth equalities, one of which being

$$\forall \rho_\mathrm{N} > 0, \quad r_\mathrm{N} - \max(r_\mathrm{N} - \rho_\mathrm{N} y, 0) = 0. \tag{2}$$

Briefly said, the set of points solution to Equation (1) and Equation (2) is the same [1, 2]. Also, with the considered parameterization, Coulomb's friction classically says the following: given a closed contact in the normal direction,

$$\begin{cases} \dot{u}_\mathrm{T} = 0 & \Longrightarrow \quad |r_\mathrm{T}| \leq \mu r_\mathrm{N} \\ \dot{u}_\mathrm{T} \neq 0 & \Longrightarrow \quad |r_\mathrm{T}| = \mu |r_\mathrm{N}| \text{ and } \exists \alpha \geq 0 \,|\, r_\mathrm{T} = -\alpha \dot{u}_\mathrm{T} \end{cases} \tag{3}$$

where μ is the coefficient of friction. Among others, the above condition can equivalently be recast into the nonsmooth equality [2, 7]

$$\min(|\dot{u}_\mathrm{T}|, \mu r_\mathrm{N} - |r_\mathrm{T}|) = 0 \tag{4}$$

or

$$\forall \rho_\mathrm{T} > 0, \quad |\dot{u}_\mathrm{T}| - \max(|\dot{u}_\mathrm{T}| - \rho_\mathrm{T}(\mu r_\mathrm{N} - |r_\mathrm{T}|), 0) = 0. \tag{5}$$

Newton's impact law
Depending on the context of the investigation, an impact law $\dot{y}^+ = -e\dot{y}^-$ relating the pre- and post-impact velocities, $\dot{y}^-$ and $\dot{y}^+$ respectively, through a coefficient of restitution $e \in [0, 1]$ might be required for the well-posedness of the formulation since Equation (1) alone might not guarantee the uniqueness of the solution [4]. The idea is to test whether a penetration between the bodies in contact has occurred and then enforce the unilateral contact conditions at the velocity level with the above impact law inserted. Altogether, this reads [2]:

$$\begin{cases} y < 0 & \Longrightarrow \quad r_\mathrm{N} \geq 0, \quad \dot{y}^+ + e\dot{y}^- \geq 0, \quad r_\mathrm{N} \cdot (\dot{y}^+ + e\dot{y}^-) = 0 \\ y \geq 0 & \Longrightarrow \quad r_\mathrm{N} = 0 \end{cases} \tag{6}$$

which can be recast into the single equality

$$\forall \rho_{\mathrm{N}}, \quad (\operatorname{sign} y - 1)(r_{\mathrm{N}} - \max(r_{\mathrm{N}} - \rho_{\mathrm{N}}(\dot{y}^{+} + e\dot{y}^{-}), 0) + (\operatorname{sign} y + 1) r_{\mathrm{N}} = 0 \tag{7}$$

with the convention sign $y = 1$ if $y \geq 0$ and sign $y = -1$ otherwise.

Governing equations

Given the geometric nonlinearity induced by the action of the spring on the mass, we introduce the quantity

$$\gamma(x, y) = \frac{\sqrt{x^2 + (L + y)^2} - L}{\sqrt{x^2 + (L + y)^2}}. \tag{8}$$

The two coupled nonlinear and nonsmooth Ordinary Differential Equations

$$m\ddot{x} + k\gamma(x, y)x - r_{\mathrm{T}} = 0 \tag{9a}$$

$$m\ddot{y} + k\gamma(x, y)(y + L) - r_{\mathrm{N}} + mg = 0 \tag{9b}$$

together with either Equation (2) and Equation (4), or Equation (7) and Equation (4), govern the dynamics of the system considered. In Equation (9b), g is the gravity constant. Depending on the level of regularity of the targeted solution, Equation (9) might have to be read in the distributional sense.

Weighted residual formulation

It is suggested to search for periodic solutions by solving the above formulation in a weighted residual sense. All unknowns of the problem are expanded on an appropriate truncated basis of T-periodic functions with N members, commonly the Fourier basis in the Harmonic Balance Method but not necessarily, as follows:

$$x(t) = \sum_k x_k \phi_k(t), \quad y(t) = \sum_k y_k \phi_k(t), \quad r_{\mathrm{N}}(t) = \sum_k N_k \phi_k(t), \quad r_{\mathrm{T}}(t) = \sum_k T_k \phi_k(t). \tag{10}$$

Depending on the smoothness of the selected basis functions, time derivatives might either be obtained by pointwise differentiation in time or expanded on a less smooth basis and related to the differentiated quantity in a weak sense. Concerning Equation (7) which requires access to $\dot{y}^{+}$ and $\dot{y}^{-}$, a Discontinuous Galerkin scheme could be used [6]. If we decide to solve Equation (9), Equation (2) and Equation (4), the weighted residual formulation would take the following form: once the expressions of Equation (10) are inserted in the selected governing equations, find the $4N$ unknowns x_k, y_k, N_k and T_k which satisfy

$$\int_0^T \phi_k(t)(\text{Eq. (9a)})\,\mathrm{d}t = \int_0^T \phi_k(t)(\text{Eq. (9b)})\,\mathrm{d}t = \int_0^T \phi_k(t)(\text{Eq. (2)})\,\mathrm{d}t = \int_0^T \phi_k(t)(\text{Eq. (4)})\,\mathrm{d}t = 0, \quad \forall k. \tag{11}$$

The above integrals can be numerically computed using an appropriate quadrature scheme such as a Riemann sum. The resulting system of nonlinear equations can be also solved using a nonsmooth Newton solver, for instance. The proposed strategy can be seen as a very compact form of the AFT methodology [5] without regularization and shares similarities with the DLFT technique [3] which also relies on the AFT. The rate of convergence of the proposed procedure might depend on the two parameters ρ_{N} and ρ_{T}, which have to be assigned a value in the solvers.

Conclusions

The proposed formulation is very compact and involves simple implementations such as basic integral quadrature schemes and existing nonlinear solvers. Its engineering value lies in its capability to generate coarse approximations without difficulty in contrast to much more advanced time-stepping or event-driven schemes [2]. It can be extended to more elaborate mechanical systems in a straightforward fashion. However, the convergence rate in terms of the number of unknowns is expected to be low and should be investigated with great care.

References

[1] Lu T. and Legrand M. (2021), Nonsmooth Modal Analysis via the Boundary Element Method for One-Dimensional Bar Systems. *Nonlinear Dynamics* 10.1007/s11071-021-06994-z

[2] Acary V. and Brogliato B. (2008), Numerical Methods for Nonsmooth Dynamical Systems: Applications in Mechanics and Electronics, Springer 10.1007/978-3-540-75392-6

[3] Nacivet S., Pierre C., Thouverez F. and and Jézéquel, L. (2003) A dynamic Lagrangian frequency–time method for the vibration of dry-friction-damped systems, *Journal of Sound and Vibration* 10.1016/S0022-460X(02)01447-5

[4] Moreau J.J. (2004), An introduction to Unilateral Dynamics. Novel Approaches in Civil Engineering. *Lecture Notes in Applied and Computational Mechanics* 10.1007/978-3-540-45287-4_1

[5] Cameron T.M. and Griffin J.H. (1989), An Alternating Frequency/Time Domain Method for Calculating the Steady-State Response of Nonlinear Dynamic Systems, *Journal of Applied Mechanics* 10.1115/1.3176036

[6] Schindler T., Rezaei S. Kursawe J. and Acary V. (2015), Half-explicit timestepping schemes on velocity level based on time-discontinuous Galerkin methods, *Computer Methods in Applied Mechanics and Engineering* 10.1016/j.cma.2015.03.001

[7] Xuewen L., Soh A.-K. and Wanji C. (2000), A new nonsmooth model for three-dimensional frictional contact problems, *Computational Mechanics* 10.1007/s004660000202