Synchronization of a Self-Excited Inertia-Wheel Pendula Array

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Summary. We investigate synchronization of a self-excited inertia wheel pendula array. The dynamical system exhibits asymptotically stable equilibria, periodic limit cycle oscillations, and non-stationary rotations. The analysis reveals that synchronous periodic oscillators are in-phase whereas quasiperiodic oscillators are out-of-phase. Furthermore, the non-stationary rotations exhibit combinations of oscillations and rotations of the individual elements which are asynchronous.

Introduction

Self-excited synchronous oscillations in multibody dynamical systems have been documented since the middle of the seventeenth century with the observation of Christiaan Huygens that two pendulum clocks hanging from a common flexible support swung together periodically approaching and receding in opposite motions[1]. Examples of synchronization in rigid-body and continuous dynamical systems have been documented for coupled mechanical metronomes[2], coupled pendula suspended from a moving beam[3], and a nano mechanical cantilever array[4]. We examine the complexity of coexisting synchronous and asynchronous self-excited oscillations in an array of three planar pendula augmented with rotating inertia wheels governed by a linear feedback mechanism. We formulate the dynamical system using a Lagrangian approach (Fig. 1 left). A linear stability map analysis of the zero equilibria yields a transition from an asymptotically stable region (Fig. 1 right-red) to a region of self-excited oscillations (Fig. 1 right-blue), culminating with a region of rotations (Fig. 1 right-white). We note that the saddle-node bifurcation $(\Gamma_3 = 0)$ for both a stationary array $(x \in \mathbb{R}^5)$ is identical to that of a moving array $(x \in \mathbb{R}^{11})$. However, the Hopf bifurcation between a stable zero equilibria and periodic oscillations reveals a slightly larger region of self-excited limit cycles (Fig. 1 right-dashed).

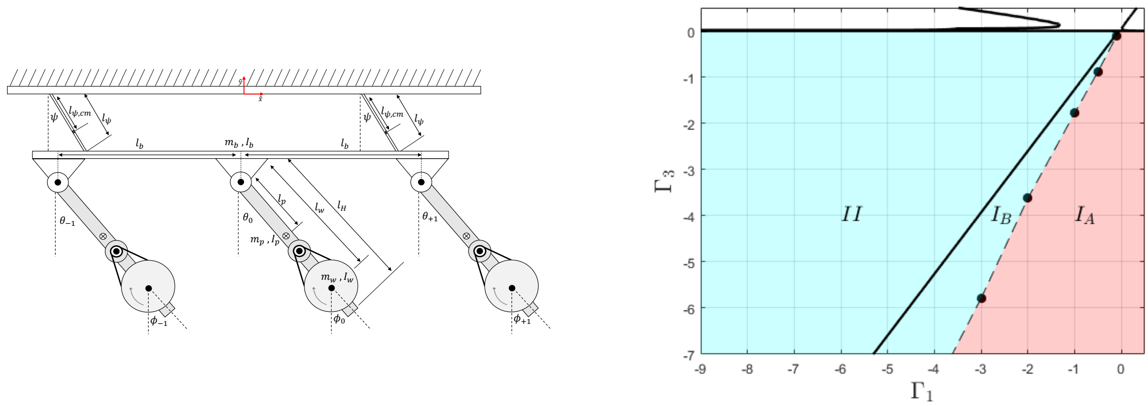

Figure 1: Definition sketch (left), stability map (right) of the inertia wheel feedback gains $\big(\Gamma_3(\Gamma_1)\big)$ for a stationary (solid) and moving (dashed) array

Results

By enforcing the analytical constraints[7] we find the Hopf threshold (Fig. 1 right-solid) of the stationary array. The Hopf threshold of the moving pendulum array is obtained numerically for varying values of the gain Γ_1 (Fig. 1 right-circles). Area I_A yields asymptotic stability for the moving array, and area I_B together with I_A yields asymptotic stability for the stationary array. We simulate the dynamics of the array for gain values Γ_3 both near and far from the Hopf threshold $\Gamma_{3H}(\Gamma_{1H})$ for an arbitrary constant gain parameter $\Gamma_1 = const$. For gains Γ_3 near the Hopf threshold we obtain periodic motion of the array elements which reveals in-phase synchronization (Fig. 2 left). As the gain Γ_3 increases we obtain quasiperiodic dynamics of the individual elements and out-of-phase synchronization (Fig. 2 center). The quasiperiodic motion leads to asynchronous chaotic oscillations (Fig. 2 right). The behavior of the dynamical system was examined through Poincaré maps (Fig. 2 bottom) portrayed by the system conjugate momenta $(p_\psi, p_{\theta i}, p_{\phi i})$ and sampled every positive zero crossing of the central inertia wheel velocity which is bounded.

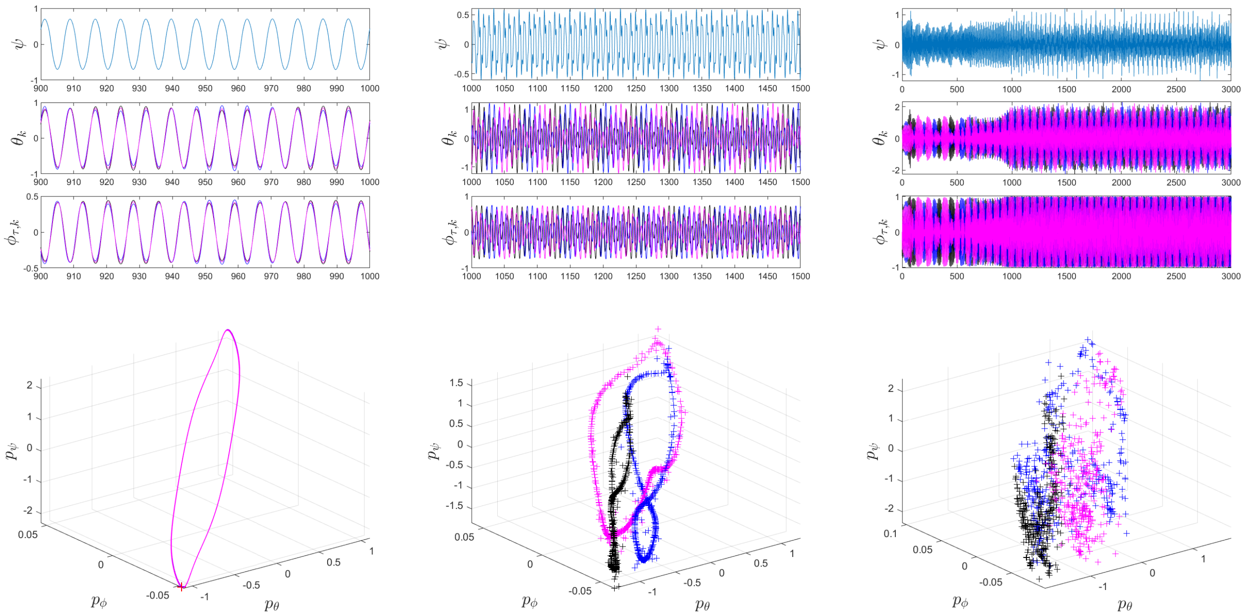

Figure 2: Time histories (upper) and conjugate momenta Poincaré maps (lower) for an in-phase periodic response (left) and an out-of-phase quasiperiodic response (center), and asynchronous chaotic oscillations (right)

Discussion

Non-stationary rotations occur first in the periphery pendula (Fig. 3 left), while the center pendulum exhibits chaotic oscillations. After a threshold gain value $\hat{\Gamma}_{3R}$ all three pendula rotate while the base oscillates chaotically (Fig. 3 center). The chaotic oscillations of the base culminate with rotations of all system elements (Fig. 3 right). We note that the linear feedback governing inertia wheel dynamics was synchronized with the pendula array periodic (out-of-phase) and quasiperiodic/chaotic (in-phase) oscillations, the non-stationary rotations were found to be asynchronous.

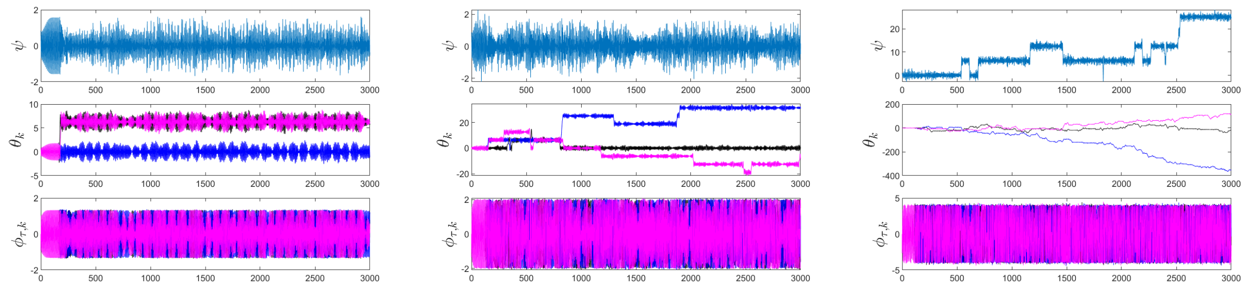

Figure 3: Time histories for non-stationary rotations combined with oscillations of the array elements (left) and (center), and non-stationary rotations of all elements (right)

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