Reduced-order Modeling from Experimental Data via Spectral Submanifolds

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Summary. Reduced-order modeling is among the leading theoretical and computational challenges for data concerning nonlinear systems in mechanics, ranging from structures and fluid flows, to their interaction and other multi-physics problems. Data-driven model reduction methods are well-established for linear dynamical systems, while available approaches for nonlinear systems often reveal to be sensitive in the identified parameters, and to be limited in prediction capabilities. With this contribution, we present an approach based on the theory of spectral submanifolds, which captures explicit nonlinear models from data. Without specific assumptions on the type of observables or the kind of measurements, our method identifies nonlinear models that exhibit the footprint of geometric nonlinearities or nonlinear damping in the observed dynamics. Our reduced-order models, which are trained on decaying vibrations data, are also capable to accurately predict forced-responses of the nonlinear dynamical system. We show the performances of our algorithm on measurement data of oscillations in structural or fluid dynamics.

Introduction

In the context of data-driven reduced order modeling, the most common approaches in the literature are Principal Orthogonal Decompositions (POD) followed by Galerkin projections [1] or Dynamic Mode Decomposition (DMD) [2]. The former method, however, needs the knowledge of the governing equations of motion to retrieve the reduced dynamics, while DMD is purely data-driven, but it cannot capture essentially nonlinear (or *non-linearizable*) dynamics [3], as, for example, transitions between equilibrium states or nonlinear frequency responses of structural vibrations. Available approaches from machine learning tend to not be robust or easy to handle for extrapolation or prediction [4]. In this contribution, we present an approach based on the recent theory of spectral submanifolds (SSMs) [5] that can extract reduced-order models from generic observables capitalizing on normal forms.

Results and discussion

Our method is a two-step procedure, whose details are described in [7]. After having embedded the data in a suitable observable space (either by using Whitney or Takens-type embedding, depending on available measurements), we perform data-driven dimensionality reduction by modeling the SSM geometry. Our reduced coordinates are the projection to the modal subspace tangent to the SSM at the equilibrium, and the nonlinearities are described via polynomials. From these arbitrary coordinates, we then seek the reduced dynamics in normal form by minimizing the conjugacy error among data, and, for oscillatory problems, the general normal form related to m linearized modes of the system reads

$$\begin{array}{l}\dot{\rho}_j = -\alpha_j(\boldsymbol{\rho},\boldsymbol{\theta})\rho_j,\\ \dot{\theta}_j = \omega_j(\boldsymbol{\rho},\boldsymbol{\theta}),\end{array} \qquad j=1,2,...,m, \qquad m\geq 1, \qquad \boldsymbol{\rho}=(\rho_1,\rho_2,...\rho_m), \qquad \boldsymbol{\theta}=(\theta_1,\theta_2,...\theta_m). \tag{1}$$

The maps α_j and ω_j are the nonlinear continuations of linearized frequency and damping, identifying how dissipation and frequency change with respect to the normal form modal amplitudes $\boldsymbol{\rho}$ and eventually phases $\boldsymbol{\theta}$, whose latter dependence is only showing up in internally resonant systems. We also remark that our modeling approach when related to multiple non-resonant modes do not assume the modes to be uncoupled. We use (1) to study the dynamics and make predictions for eventual forced responses, and afterwards, using the SSM geometry, we can trace back normal form amplitudes to physical observed quantities. To show the performance of our method in capturing the nonlinear behavior in systems with different physics, we consider an example in fluid-structure interaction and one featuring structural vibrations.

We first consider the liquid sloshing example from [6, 7] depicted in Fig. 1(a), where measurements are carried out via a laser system and the excitation is provided by moving the platform onto which the tank sits. We set our observable to be horizontal position of the water center of mass and we focus on the slowest system mode. We train our SSM-based model from the resonance decay data shown in Fig. 1(b) using cubic nonlinearities, showing already good accuracy in reconstructing test trajectories. The model identifies the nonlinear backbone curves, characterizing damping $\alpha(\rho)$ and frequency $\omega(\rho)$, and is then used to compute frequency responses, which follows the reduced dynamics

$$\dot{\rho} = -\alpha_0\rho - \beta\rho^3 + f\sin(\psi), \qquad \dot{\psi} = \omega_0 + \gamma\rho^2 - \Omega + \frac{f}{\rho}\cos(\psi), \tag{2}$$

where Ω is the forcing frequency, f the forcing amplitude and $\psi = \theta - \Omega t$ the phase lag. Forced periodic solutions can be sought in closed form from eq. (2). In Fig. 1(c), we show these forced responses, where the dots are experimental measurements for different forcing frequency and forcing amplitude values, while solid lines are predictions from (2), after proper calibration for finding the normal form forcing amplitude f. Even tough our model has been trained only on unforced data, it exhibits great accuracy in predicting forced responses (thanks to our detailed modeling of softening and of nonlinear damping), also for amplitudes being higher with respect to those of training data.

Another example is the two-beam system of Fig. 1(d). The first two modes of this assembly feature a $1:2$ internal resonance and system nonlinearities are due to weak frictional contact happening at the joint between the inner beam and

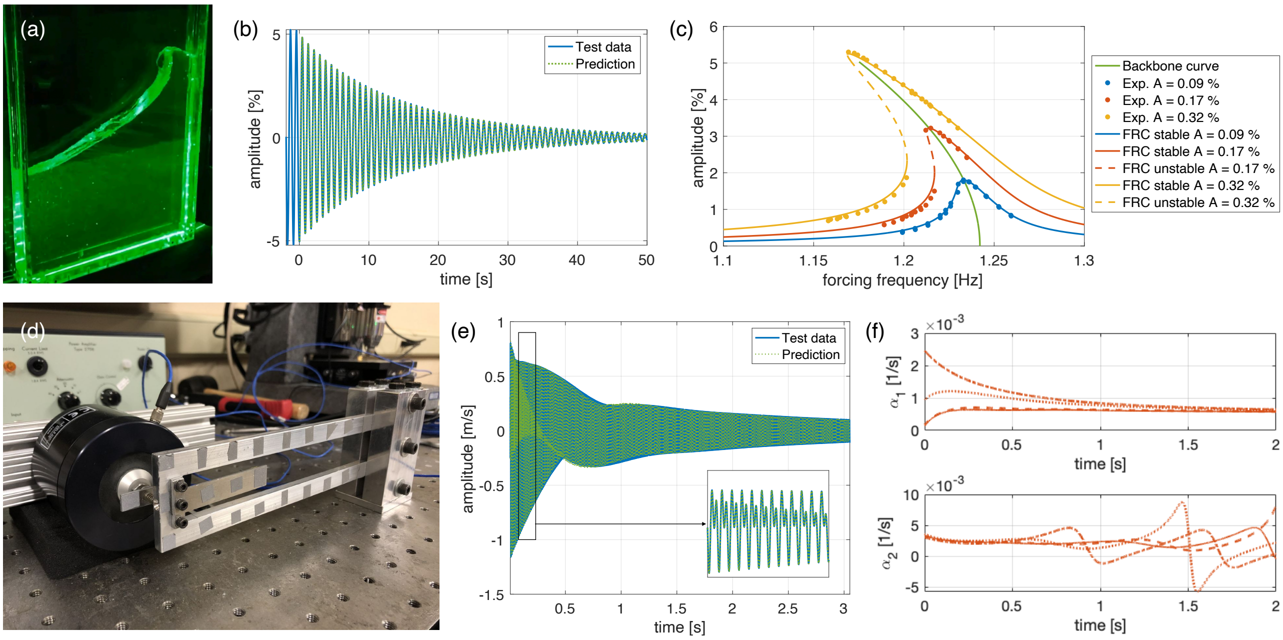

Figure 1: (a) Photo of measurements of liquid sloshing in a tank. (b) Decaying oscillations released from resonant quadrature forcing with model predictions. The amplitude is the horizontal displacement of the center of mass of the water inside the tank, expressed in percentage after normalization with respect to the tank width. (c) Analytical Forced Response Curves (FRCs), the damped backbone curve (blue solid line) and experimental measurements for different forcing amplitudes and frequencies. (d) Picture of the resonant tester structure. (e) Decaying resonant oscillations excited via hammer impact of the inner beam along with model predictions. The amplitude is the velocity of the inner beam tip. (f) Nonlinear damping trends for the slow α_1 and fast α_2 modes along some decaying trajectories.

the external one, which is clamped to the ground on the other side. Decaying vibrations are excited using hammer impacts on different locations of the inner beam and our observable is the inner tip velocity measured via laser scanner vibrometry. Due to internal resonance, transients show two dominant frequencies, cf. Fig. 1(e), as they very quickly converge to the slow four-dimensional SSM, i.e., related to the two slow system modes. In this case, our method automatically detects the internal resonance from data and it identifies a cubic order model with the specific form

$$
\begin{aligned}
\dot{\rho}_1 &= -\,\alpha_{0,1}\rho_1 - \beta_{11}\rho_1^3 - \beta_{12}\rho_2^2\rho_1 - \rho_1\rho_2\left(\sigma_{11}\cos\psi - \sigma_{12}\sin\psi\right), \\
\dot{\rho}_2 &= -\,\alpha_{0,2}\rho_2 - \beta_{21}\rho_1^2\rho_2 - \beta_{22}\rho_2^3 - \rho_1^2\left(\sigma_{21}\cos\psi + \sigma_{22}\sin\psi\right), \\
\rho_1\dot{\theta}_1 &= +\,\omega_{0,1}\rho_1 + \gamma_{11}\rho_1^3 + \gamma_{12}\rho_2^2\rho_1 + \rho_1\rho_2\left(\sigma_{11}\sin\psi + \sigma_{12}\cos\psi\right), \\
\rho_2\dot{\theta}_2 &= +\,\omega_{0,2} + \gamma_{21}\rho_1^2\rho_2 + \gamma_{22}\rho_2^3 + \rho_1^2\left(\sigma_{22}\cos\psi - \sigma_{21}\sin\psi\right),
\end{aligned}
\tag{3}
$$

where $\psi = \theta_2 - 2\theta_1$ is the internal phase shift. Our data-driven model is able to reconstruct trajectories test with an average 1.2 % error, as in the example of Fig. 1(e). In particular, the damping of the fast mode undergoes consistent variation as shown in Fig. 1(f), becoming also negative for some times, since the fast mode tries to absorb energy from the slow mode. Additional details of this example are reported in [8].
Our data-driven approach is implemented on the open-source MATLAB® package `SSMLearn`, which is to be released soon. Other than the source code, the repository contains the data sets discussed in this contribution and the live-scripts with their analysis, and also additional worked examples.

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