An efficient method to obtain the response PDF of nonlinear stochastic dynamical systems

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Summary. We present an efficient formulation of the path integration method to approximate the response probability density functions (PDF) of nonlinear stochastic differential systems. We obtain the time-dependent PDF at the next time point by solving the Chapman-Kolmogorov (CK) equation, where we integrate the probability density function (TPDF) of the transition from one state to another (TPDF), for all possible states, weighted by the PDF at the current time. We iteratively evaluate the CK equation via a partitioned formulation of the path integration method: the PDF is represented as an interpolated function, the TPDF from one state to the other is approximated using a numerical scheme, and the integral is evaluated using the Gauss-Legendre quadrature. We record the integration process in a so-called step matrix and transform the evaluation of the CK equation to a matrix-vector multiplication. We demonstrate that this approach increases the performance of the path integration method compared to previously established approaches and analyse the accuracy of different interpolation and time-stepping methods.

Introduction

The probability density function (PDF) is an important tool to investigate the response statistics of dynamical systems subjected to noise excitation. When the dynamical system is represented as a stochastic differential equation (SDE), a commonly used method to determine the response PDF is to solve the corresponding Fokker-Planck or Kolmogorov forward equation. This partial differential equation rarely has an exact analytical solution, and thus in most cases, we have to use a numerical approximation method to solve it. The simplest method to obtain the PDF is through time-domain Mone-Carlo simulations when we numerically integrate the SDE in time and use the realised trajectories to approximate the PDF.

Another method to obtain the PDF of an SDE is the path integration (PI) method. Here we have to solve the Chapman-Kolmogorov (CK) equation, formulated to describe the law of total probability for dynamical systems. The PI method has proven to provide an accurate estimate of the time evolution of the PDF of a dynamical system; however, the CPU time required to compute a PDF is still an issue. The computation time is a critical problem to address if the dimensionality of the state space of the investigated dynamical system is high or when we want to obtain the steady-state PDF for a slowly converging system. In previous formulations of the PI method [1, 2, 3, 4] the CK equation was directly solved in each time step, even in the case of time-invariant or time-periodic systems. There are works [5] that utilise FFT to speed up the time it requires to evaluate the CK equation for each time step; however, the CK equation is still evaluated at each time step.

In this work, we provide a modular approach to solve the CK equation and analyse the effect of different interpolation methods on the accuracy and the performance of the approximation of the PDF delivered by the numerical solution of the CK equation. Furthermore, we transform the process of evaluating the CK equation to a matrix multiplication which significantly speeds up the computation of PDFs of time-invariant and time-periodic systems.

Chapman-Kolmogorov Equation

We consider stochastic differential equations in the form

$$\mathrm{d}\mathbf{x}(t) = \mathbf{f}(\mathbf{x}, t)\mathrm{d}t + \mathbf{g}(\mathbf{x}, t)\mathrm{d}\mathbf{W}_t, \tag{1}$$

where $\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_d \end{bmatrix}^\top$ is the $\mathbb{R}^d$-valued stochastic state variable, $\mathbf{W}_t$ is the $\mathbb{R}$-valued Wiener process (Brownian motion), $\mathbf{f} : \mathbb{R}^d \times [0, T] \mapsto \mathbb{R}^d$, $\mathbf{g} : \mathbb{R}^d \times [0, T] \mapsto \mathbb{R}^d$. We can assume, that the diffusion term $\mathbf{g}$ has only $d - k + 1$ nonzero terms:

$$g_i(\mathbf{x}, t) \equiv 0 \quad \text{for} \quad i < k \leq d. \tag{2}$$

We use the Chapman-Kolmogorov equation to obtain the probability density function $p(\mathbf{x}, t)$ in discrete times t_n and t_{n+1}:

$$p(\mathbf{x}, t_{n+1}) = \int_{\mathbb{R}^d} p(\mathbf{x}, t_{n+1}|\mathbf{x}_0, t_n)p(\mathbf{x}_0, t_n)\mathrm{d}\mathbf{x}_0. \tag{3}$$

As the solution of the Chapman-Kolmogorv equation is not available in an analytical form for a general smooth $\mathbf{f}$ and $\mathbf{g}$, we need to discretise (3) using the path-integration approach. We approximate the transitional probability density function (TPDF) $p(\mathbf{x}, t|\mathbf{x}_0, t_0)$ with the help of a numerical stepping scheme, interpolate the probability density function (PDF) $p(\mathbf{x}, t)$, and evaluate of the integral with the help of the Gauss-Legendre quadrature.

We record the whole calculation process in a step matrix $\mathbf{S}_n$ and substitute the evaluation of the integral (3) with a matrix-vector multiplication:

$$\mathbf{p}_{n+1} = \mathbf{S}_n \mathbf{p}_n. \tag{4}$$

In $\mathbf{p}_n$ we record the interpolation values describing the PDF $p(\mathbf{x}, t)$. In case the SDE in (1) is time-invariant or time-periodic, this approach will lead to a very performant method to solve (3).

Accuracy and Performance

Due to the partitioned formulation of the PI method we are able to separately investigate the effect of the time stepping method used for the approximation of the TPDF $p(\mathbf{x}, t_{n+1}|\mathbf{x}_0, t_n)$ and the interpolation method of the PDF $p(\mathbf{x}, t_n)$ on the accuracy of the approximation. We analyse the different time stepping and interpolation methods through the response PDF of a cubic oscillator:

$$\ddot{x}(t) + 0.3\dot{x}(t) - x(t) + 0.25x(t)^3 = \sqrt{0.075}\,\xi(t). \tag{5}$$

The steady-state PDF $p_{\mathrm{st}}(x, \dot{x})$ of (5) is approximated with the help of the path integration method and is compared with the true solution [6]. To characterise the accuracy of the PI method we use ε_1 that is obtained by integrating the absolute error between the true and approximated steady state PDF $p_{\mathrm{st}}(x, \dot{x})$ for the all the states $x, \dot{x} \in \mathbb{R}$.

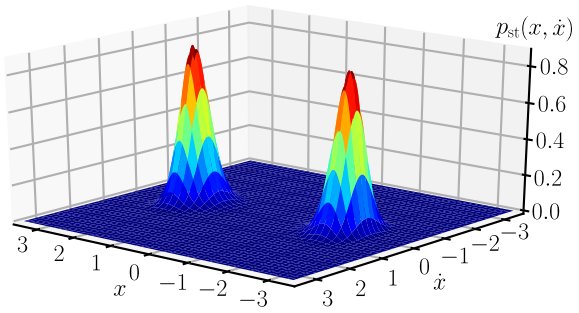

Figure 1: Steady state PDF $p_{\mathrm{st}}(x, \dot{x})$ of the cubic oscillator (5)

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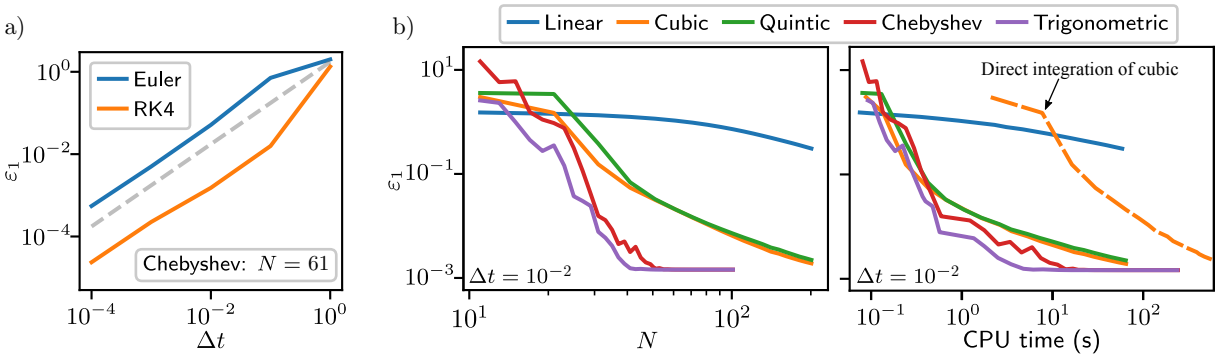

Figure 2: Error ε_1 as a function of (a) time step Δt and (b) number of interpolation nodes N / CPU time of the approximated steady state response PDF $p_{\mathrm{st}}(x, \dot{x})$ of the cubic oscillator (5)

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We demonstrate that our new formulation allows accurate computation of the PDF of noise-driven nonlinear dynamical systems significantly faster than previous formulations. Additionally, we show that the increased performance of the PI method allows the parametric analysis of a vibro-impact energy harvesting device that is used to harvest energy from noisy ambient vibrations.

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