Entrainment of Self-Organized Synchronized States in Delay-Coupled Oscillators

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Summary. This work presents how synchronized states that self-organize in networks of mutually coupled oscillators can be entrained by an external reference oscillator. A second-order Kuramoto model with time-delayed coupling is used to predict the phase-differences in the network and the stability of the entrained states. The model predictions are then verified by experimental measurements.

Introduction

Synchronization is important for well-defined and concerted operations and provides the means to establish and keep time-synchronization in networks of spatially distributed clocks [1]. In technical applications synchronization is usually achieved by entraining an oscillator with a periodic signal of a high quality reference oscillator [2]. In natural systems self-organized synchronization is prevalent and based on a mutual coupling between, e.g., cellular oscillators. In this manuscript we study the complex dynamics in finite system of inert, mutually delay-coupled oscillators when subject to an external forcing. The theoretical results for the minimal case, where a reference entrains a network of two mutually delay-coupled oscillators, are verified by experiments with electronic phase-locked loop (PLL) oscillators [4]. We discuss the experimental results taking into account the basins of attraction of the synchronized states that are studied.

Model for Studying the Entrainment of Networks of Mutually Delay-Coupled Oscillators

The dynamics in networks of inert, delay-coupled oscillators can be studied using a second-order Kuramoto model [3, 4]

$$m_k\,\ddot{\theta}_k(t) + \dot{\theta}_k(t) = \omega_k + \frac{K_k}{n_k}\sum_{l=1}^{N} c_{kl}\, h\left(\theta_l(t-\tau_{kl}) - \theta_k(t)\right), \tag{1}$$

where $k = 1, \ldots, N$ indexes the N oscillators, $\omega_k \in \mathbb{R}$ denotes the intrinsic frequencies, $h(\cdot)$ a periodic coupling function, $K_k \geq 0 \in \mathbb{R}$ the coupling strength, $m_k \geq 0 \in \mathbb{R}$ an inertial parameter, $n_k \geq 0 \in \mathbb{N}_0$ the number of inputs of oscillator k, $\theta_i(t) \in S^1$ for $i = \{k, l\}$ the phases of the oscillators' output signals with $\dot{\theta}(t)$ and $\ddot{\theta}(t)$ denoting their first and second time derivatives, $c_{kl} = \{0, 1\}$ the components of the network's adjacency matrix, and $\tau_{kl} \in \mathbb{R}$ denotes the cross-coupling time delays. The ansatz to study synchronized states and their linear stability is

$$\theta_k(t) = \Omega t + \beta_k + \epsilon q_k(t), \tag{2}$$

where Ω denotes the global frequency of a synchronized state, $\epsilon q_k(t)$ a small perturbation ($\epsilon \ll 1$), and β_k a phase-offset.

Network of mutually delay-coupled oscillators: frequency and phase differences of synchronized states
Using the ansatz (2) in Eqs. (1) and expanding $h(\cdot)$ to first order with respect to ϵ, we obtain the properties of synchronized states from $\mathcal{O}(\epsilon^0)$: $\Omega = \omega_k + K\, h\left(-\Omega\tau_{kl} + \beta_{kl}\right)$, where $\beta_{kl} = \beta_k - \beta_l$, e.g., equal to 0 or π for identical oscillators. The linear stability of these states depends on the the oscillator's parameters and the properties of the synchronized states [4].

Individual oscillator entrained by a reference: phase difference with respect to reference oscillator
Here, the frequency of the synchronized state is determined by the reference $\Omega = \omega_R$. The phase difference is $\beta = -h^{-1}\left[(\omega_R - \omega)/K\right] - \omega_R\tau$. The stability depends only on the detuning of the frequencies and the coupling strength [5].

Networks of mutually delay-coupled oscillators entrained by a reference: phase differences
Here, we consider a network of heterogeneous oscillators, making the ansatz (2) for $\Omega = \omega_\mathrm{R}$. The entrainment is accounted for by assigning the reference oscillator with $k = 1$ and setting $c_{1l} = 0\,\forall\, l$. The $N-1$ phase differences can then be obtained from

$$\omega_R = \omega_k + \frac{K_k}{n_k}\sum_{l=1}^{N} c_{kl}\, h\left[-\omega_R\tau_{kl} - \beta_{kl}\right]. \tag{3}$$

Example of a Minimal Entrained Network and Experimental Verification

We study a network of two mutually coupled oscillators one of which is forced by an external reference, see Fig. 1. From Eq. (3) the phase differences between the mutually coupled oscillators $\beta_{23} = \omega_R\tau_{32} + h^{-1}\left[(\omega_R - \omega_3)/K_3\right]$ and that to the reference $\beta_{R2} = \omega_R\tau_{R2} + h^{-1}\left[2\left(\omega_R - \omega_2\right)/K_2 - h\left[-\omega_R\tau_{23} - \beta_{23}\right]\right]$ are obtained. The theoretical predictions and the experimental results β_{23} and β_{R2} of the synchronized states are shown as a function of the reference frequency in Fig. 2c. The range of reference frequencies for which synchronized states are linearly stable is shown in green. In experiments however, we only find stable synchronized states in a smaller range than predicted. This can be explained by the basins of attraction obtained from time-series simulations shown in Figs. 2a, 2b. They show the basins for different initial phase histories for $t \in [-\tau, 0]$. In Fig. 2a all oscillators are initially free-running, as can be realized in the experimental setup.

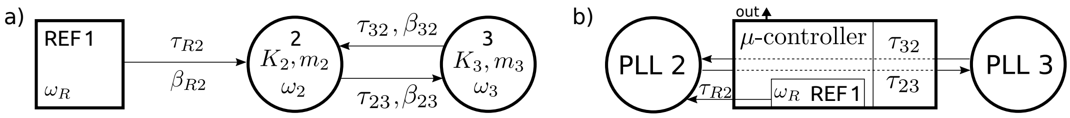
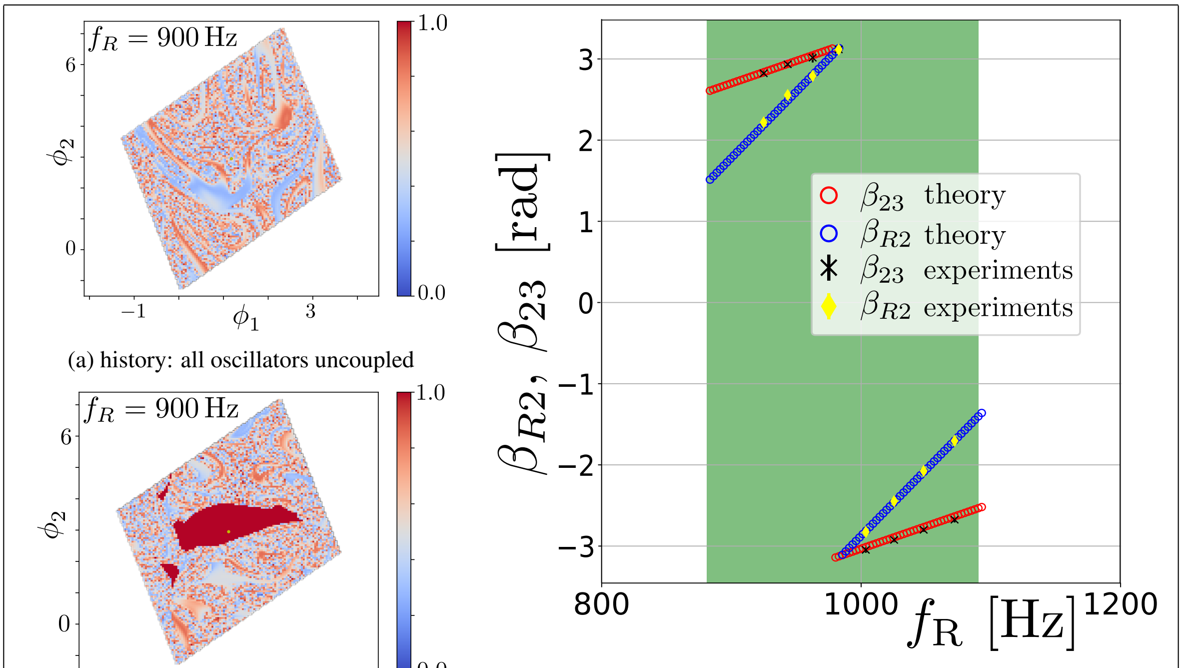


Figure 1: a) Sketch of entrainment of a network of two mutually delay-coupled oscillators. b) The experimental setup consists of a microcontroller that organizes the delay-coupling between the two mutually coupled PLLs and the entrainment by a virtual reference derived from its internal clock. For details about the experimental setup see reference [4].

In Fig. 2b, the simulation starts in the entrained synchronous state, similarly to linear stability analysis. Their x- and y-axis represent the phase differences $\phi_1 = \theta_3(0) - \theta_2(0)$ and $\phi_2 = \theta_3(0) - \theta_1(0)$ between the oscillators at $t = 0$, respectively. The color encodes the asymptotic value of the order parameter for any combination $(\phi_1,\ \phi_2)$, obtained from a time series simulation of the oscillator network using the Eqs. (1). The order parameter has been modified such that it is equal to one if the phase configuration of the entrained synchronous state under investigation is achieved [3]. From Fig. 2a it can be understood that the basin of attraction has zero volume close to the boundaries of linear stability. Hence, we cannot find entrained synchronous states with the experimental setup (Fig. 1b) that always starts from an initially free-running state.



(a) history: all oscillators uncoupled

(b) history: entrained synchronized state

(c) The phase differences β_{23} (red) and β_{R2} (blue) versus the frequency of the reference. The green area denotes linearly stable states. Experimental results shown by black stars and yellow diamonds.

Figure 2: Parameters: $\omega_2 = 1004{\cdot}2\pi$ Hz, $\omega_3 = 996{\cdot}2\pi$ Hz, $K_2 = 423{\cdot}2\pi$ Hz, $K_3 = 408{\cdot}2\pi$ Hz, and $\tau_{R1} = \tau_{12} = 0.265$ s.

Conclusions

This work studies the entrainment of self-organized synchronous states. Our phase oscillator model takes into account time-delays in the coupling and inert oscillator response. It predicts the properties of entrained synchronous states as verified here by experimental results obtained from electronic oscillators. Self-organized synchronous states can be viewed as an effective oscillator that has emerged over a network of mutually coupled oscillators. This effective oscillator is characterized by its quiescent frequency Ω and the frequency range within which it can lock to a reference. Both properties affect the linear stability of entrained synchronous states and depend on the delays within the mutually coupled oscillators. The delay between reference and network only affects the phase differences of the entrained synchronized states.

References

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