Nonlinear damping laws preserving the eigenstructure of the momentum space for conservative linear PDE problems: a port-Hamiltonian modelling

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Summary. Application to morphing in sound synthesis with the mutation of damping material properties leads us to introduce a class of nonlinear damping models operating on the momentum equation of the Hamiltonian formulation of a conservative mechanical PDE; the modal decomposition of the original linear vibrating structure is useful to analyze the preserved geometric features.

Initial conservative mechanical problem

We consider *linear conservative mechanical* systems, the solutions of which admit an eigen-decomposition. Typically, they can be finite-dimensional (mass-stiff) ODE systems, or infinite-dimensional continuous problems on a *bounded space domain* with *homogeneous boundary conditions*, governed either by the PDE (1) or by Hamiltonian descriptions (2a-2b).

PDE description

The models under consideration have the form

$$M(z)\ddot{w}(z,t)+\mathcal{K}(z)w(z,t)=f(z,t)\ \text{ for all }\ (z,t)\in\Omega\times\mathbb{R}_+, \tag{1}$$

with zero initial conditions, where z and t denote the space and time variables, Ω a bounded space domain, w a displacement, f an external force, M a mass matrix and $\mathcal{K}$ a structured differential stiffness operator. Technically, we assume that $M(z)$ is a symmetric uniformly positive definite matrix ($M\in L^\infty(\Omega,S_n^+)$, $\varepsilon\,\mathrm{Id}\leq M(z)\leq \|M\|_{L^\infty}\,\mathrm{Id}$), that $\mathcal{K}(z)$ is a symmetric positive differential operator such that $M^{-1}\mathcal{K}$ defines a self-adjoint operator on a Hilbert space $\mathbb{H}$. In practice, $\mathcal{K}$ can be a spatial operator with classical (e.g. Dirichlet or Neumann) homogeneous boundary conditions (see example).

Hamiltonian description

We also assume that this model admits a Hamiltonian description (including the excitation). The mechanical state $X=[q,p]^\intercal$ is composed of a configuration variable $q(z,t):=\mathcal{J}_{qp}(z)\,w(z,t)$ (that typically encodes a geometrical deformation) and the momentum $p(z,t):=M(z)\,\dot{w}(z,t)$. The Hamiltonian is $H\big(X=[q,p]^\intercal\big)=\frac{1}{2}\int_\Omega\big(p(z)^\intercal L_p(z)p(z)+q(z)^\intercal L_q(z)q(z)\big)\,\mathrm{d}z$ with $L_p:=M^{-1}$ and L_q symmetric and uniformly positive, so that its variational derivative is $\delta_X H(X)=\mathcal{L}X$ with $\mathcal{L}:=\mathrm{diag}(L_q,L_p)$. The governing equation is

$$\partial_t X=\mathcal{J}\,\delta_X H(X)+\begin{bmatrix}0\\1\end{bmatrix}f \qquad \text{with } \mathcal{J}(z)=\begin{bmatrix}0 & \mathcal{J}_{qp}(z)\\ -\mathcal{J}_{qp}^*(z) & 0\end{bmatrix}, \tag{2a}$$

$$v=[0,1]\,\delta_X H(X), \tag{2b}$$

where $\mathcal{J}=-\mathcal{J}^*$ is skew-symmetric ($\mathcal{J}_{pq}=-\mathcal{J}_{qp}^*$). The observation equation (2b) produces the power-conjugated variable v of excitation f, and this system is power-balanced: $\mathrm{d}H(X(\cdot,t))/\mathrm{d}t$ is the sum of the external power $\langle f(\cdot,t),v(\cdot,t)\rangle$ supplied in Ω and that incoming through boundaries (zero for the homogeneous conditions assumed in this paper) [2]. Equation (2a) relates the efforts $e=[e_q,e_p]^\intercal:=\delta_X H=[L_q\,q,L_p\,p]^\intercal)$ to the flows $f=[f_q,f_p]^\intercal:=\partial_t X=[\dot{q},\dot{p}]^\intercal$.
Note that its interpretation on the displacement variable reads

$$f_q=\mathcal{J}_{qp}e_p \quad\longrightarrow\quad (\mathcal{J}_{qp}\,\dot{w})=\mathcal{J}_{qp}\,\dot{w} \qquad \text{(kinematic concordance equation)} \tag{3}$$

$$f_p=-\mathcal{J}_{qp}^*e_q+f \quad\longrightarrow\quad (M\,\ddot{w})=-\mathcal{J}_{qp}^*\,L_q\mathcal{J}_{qp}w+f \qquad \text{(momentum balance)} \tag{4}$$

and $v=\dot{w}$ (output), so that the momentum balance corresponds to (1) with the meaningful factorization $\mathcal{K}=\mathcal{J}_{qp}^*\,L_q\mathcal{J}_{qp}$.

Example of a rectangular membrane

Consider a 2D membrane ($z=(x,y)\in\Omega=(0,X)\times(0,Y)$) with transverse displacement w [m], surface mass density $M(z)=\rho(z)>0$ [Kg/m^2], tension tensor $T_{2\times2}(z)$ [N/m] and with fixed boundaries so that $\mathcal{K}(z)=-\mathrm{div}\Big(T_{2\times2}(z)\vec{\mathrm{grad}}\Big)$ is defined on $\mathcal{D}=\{w\in H^2(\Omega)\text{ s.t. } w|_{\partial\Omega}=0\}$ in (1).
For a homogeneous membrane with constant parameters $\rho(z)=\rho_0$ and $T_{2\times2}(z)=T_0I_2$, both operators $\mathcal{K}(z)=-T_0\Delta$ and $M(z)^{-1}\mathcal{K}(z)=-(T_0/\rho_0)\Delta$ involve the standard Laplacian $\Delta:=\mathrm{div}\,(\vec{\mathrm{grad}})=\partial_x^2+\partial_y^2$ defined on domain $\mathcal{D}$.
In this simple case, $M^{-1}\mathcal{K}$ is a Riesz-spectral operator. Its point spectrum is composed of positive eigenvalues ω_{mn}^2 with $\omega_{mn}=\sqrt{\frac{T_0}{\rho_0}}\sqrt{(\frac{m\pi}{X})^2+(\frac{n\pi}{Y})^2}$ for integers $m,n\geq1$. These eigenvalues ω_{mn}^2 are associated with the eigenfunctions $e_{mn}(x,y)=\frac{2}{\sqrt{XY}}\,\sin(\frac{m\pi x}{X})\,\sin(\frac{n\pi y}{Y})$, which are orthonormal in $\mathbb{H}=L^2(\Omega)$. The dynamics $w_{mn}(t):=\langle w(\cdot,t),e_{mn}\rangle_{\mathbb{H}}$ carried by each modal space function e_{mn} is governed by (1) projected on e_{mn} ($\ddot{w}_{mn}+\omega_{mn}^2w_{mn}=\langle f,e_{mn}\rangle_{\mathbb{H}}$). Each ODE is associated with two poles $\lambda_{mn}^\pm=\pm i\omega_{mn}$ (roots of the characteristic polynomials $\mathcal{P}_{mn}(\lambda)=\lambda^2+\omega_{mn}^2$).

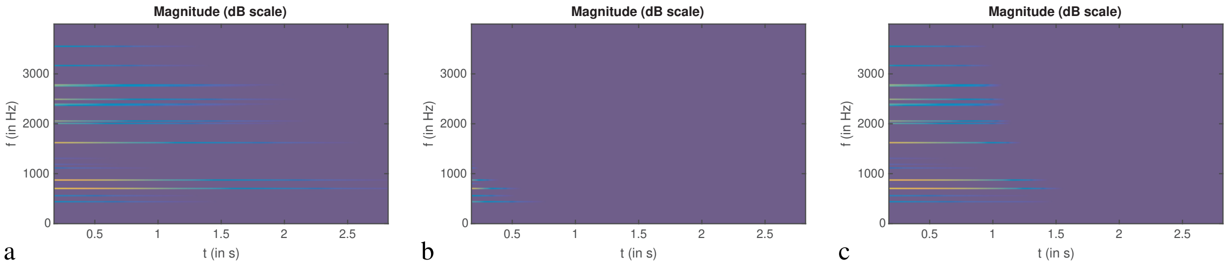

Figure 1: Spectrograms of signals produced for: (a,b) linear damping for two sets of parameters; (c) a nonlinear damping with behaviour "a" at high energies and "b" at low energies.

A natural Hamiltonian description is obtained by introducing the strain[1] $\vec{q} := \vec{\mathrm{grad}}\, w$ (meaning that $\mathcal{J}_{qp} := \vec{\mathrm{grad}}$ and $\mathcal{J}_{qp}^* = -\mathrm{div}$)), the surface momentum $p := \rho_0 \partial_t w$ and the Hamiltonian weighted by $L_p = 1/\rho_0$ and $L_q = T_0\, I_2$. Matrix $\mathcal{J}$ is then given by $\mathcal{J}(z) := \begin{bmatrix} 0 & \mathcal{J}_{qp}(z) \\ -\mathcal{J}_{qp}^*(z) & 0 \end{bmatrix} = \begin{bmatrix} 0 & \vec{\mathrm{grad}} \\ \mathrm{div} & 0 \end{bmatrix}$, and $\mathcal{J}\delta_X H = \mathcal{J}\mathcal{L} = \begin{bmatrix} 0 & \frac{1}{\rho_0}\vec{\mathrm{grad}} \\ T_0 \mathrm{div} & 0 \end{bmatrix}$ is a Riesz-spectral operator with eigenvalues $\lambda_{mn}^{\pm} = \pm i\omega_{mn}$ and eigenfunctions $E_{mn}^{\pm} := [\vec{q}_{mn} := \vec{\mathrm{grad}}\, e_{mn}, p_{mn}^{\pm} := \rho_0 \lambda_{mn}^{\pm} e_{mn}]^\intercal$. We now look for damping that preserves the kinematics (3) and the eigenstructure of the p-subspace (modes e_{mn}).

Structured damping class: linear and nonlinear models

Modifying the governing equation (2a) as $\partial_t X = (\mathcal{J} - \mathcal{R})\, \delta_X H + [0, 1]^\intercal f$ introduces some dissipation in the system if $\mathcal{R}(z) = \mathcal{R}(z)^*$ is symmetric positive.

In this paper, inspired by [1] and following previous work in [3, 4], we first propose the linear damping class built on the parameters and the structure of the initial conservative system thanks to a polynomial function $\mathbb{P}$ with positive coefficients:

$$\mathcal{L}\mathcal{R} = \Sigma\, \mathbb{P}\big((\mathcal{L}\mathcal{J})^* \, (\mathcal{L}\mathcal{J}) \big) \ \text{ with } \Sigma = \begin{bmatrix} 0_{qq} & 0_{qp} \\ 0_{pq} & I_{pp} \end{bmatrix}. \tag{5}$$

Theorem: equation (5) defines a damping class that : (i) preserves the kinematics concordance equation (3) of the original problem, between $\partial_t \vec{q}$ and $\delta_p H$; (ii) preserves the eigenstructure of the p-subspace.

The element of the proof are the positivity of operators $(\mathcal{L}\mathcal{J})^*(\mathcal{L}\mathcal{J})$ and of the coefficients of $\mathbb{P}$ for the dissipation, the selection matrix Σ that operates property (i), and the powers of $(\mathcal{L}\mathcal{J})^*(\mathcal{L}\mathcal{J})$ generated by the polynomial $\mathbb{P}$ for (ii). $\diamond$

Secondly, we generalize the previous class from linear to *nonlinear* dynamics, by making the positive coefficients of the polynomial $\mathbb{P}$ depend on the state $X(t)$ and using the new damping operator $\mathcal{R}\big(X(t)\big)$ instead of $\mathcal{R}$: indeed, passivity and properties (i-ii) still stem from the genuine structure (5).

Simulation for the homogeneous membrane

Three simulations of a membrane (see spectrograms of $\partial_t w$ in figure 1a-c) are performed for three damping polynomials $\mathbb{P}_{a,b,c}$ of the operator $(\mathcal{L}\mathcal{J})^*(\mathcal{L}\mathcal{J}) = \frac{T_0}{\rho_0}\, \mathrm{diag}(-\vec{\mathrm{grad}}\, \mathrm{div}, -\Delta)$: $\mathbb{P}_a$ of degree 0 (fluid damping) with constant coefficient; (b) $\mathbb{P}_b$ of degree 1 (fluid and structural damping) with constant coefficients; (c) an interpolation $\mathbb{P}_c = f(e)\mathbb{P}_a + (1-f(e))\mathbb{P}_b$ for e.g. the interpolation function $f(e) = \tanh\sqrt{e/e_0}$ driven by the energy signal $e(t) = H(X(t))$. As a result the damping locally behaves like (a) for energies $e \gg e_0$, like (b) for $e \ll e_0$ (including the sound extinction), building a physically-morphed sound in-between that can be interpreted as complex or mutating materials.

Conclusion

The port-Hamiltonian setting proves useful to analyze morphing strategies applied to sound synthesis, since the geometry of the underlying physics can be fully preserved during the transformation involving nonlinear damping models, which are easily parameterized using polynomials.

References

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[1] we use $\vec{q}$ to mark the 2D vector on the configuration variable.