Unified perspectives on nonlinear model reduction

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Summary. The past decade has witnessed renewed interests to nonlinear model reduction and many differently motivated techniques are proposed, including nonlinear normal modes (NNMs), direct multi-scale method (dMSM, full-basis), normal form (NF), sub-spectral manifolds (SSMs), quadratic manifolds (QM, using a mode derivative concept), low-order elimination technique (LOE). There is indeed demand for unified perspectives on the whole nonlinear model reduction matter, aiming at a better understanding of subtle connections among all these reduction methods.

In this talk, the recent low-order elimination (LOE) technique using a passive pattern concept will be first discussed, and then used to outline some unified perspectives on nonlinear model reduction, which are developed based upon two different basic problems, i.e., truncation order and truncation degree. The former refers to the common reduced dimension or number of dominant modes, while the latter refers to truncation degree of polynomials employed to approximate invariant manifold/transformation/passive pattern.

An explicit theoretical correspondence among these reduction methods will be detailed, placing in particular NNMs/dMSM/NF/LOE within a unified framework in the sense of refined finite mode truncation, which justifies various claims/observations in literature that all these refined reduction methods correct the routine/flat Galerkin (say, single-mode) truncated model.

Another unified perspective is built by focusing on various reduced-order models (ROMs) of general quadratic/cubic nonlinear structures, produced by different reduction methods with two-, three-, and four-degree truncations. It turns out that all the truncations produce valid and equivalent, but seemingly different, ROMs. Through translating invariant manifolds and nonlinear transformation terminology into low-order elimination language using passive patterns, we frame various reduction approaches within the same formulation and finally give a unified elucidation of distinct reduction methods in the sense of truncation degree.

Model truncation issues

We take a quadratic/cubic nonlinear structure as an archetypical model

$$\partial^2 w/\partial t^2 + L[w] = N_2[w] + N_3[w] + \cdots \tag{1}$$

with boundary conditions $B_0[w] = B_1[w] = 0$. Here $w(x,t)$ is the displacement, $L[\cdot]$, $N_2[\cdot]$ and $N_3[\cdot]$ are the structure's linear, quadratic and cubic spatial operators, respectively. Using single-mode truncation (*m*-th mode is assumed to dominate asymptotic dynamics), we deduce routine Galerkin model for model (1)

$$\ddot{q}_m + \omega_m^2 q_m = \langle \phi_m, N_2[\phi_m, \phi_m] \rangle q_m^2 + \langle \phi_m, N_3[\phi_m, \phi_m, \phi_m] \rangle q_m^3 + \cdots \tag{2}$$

where $\langle \phi_i, \phi_j \rangle = \delta_{ij}$, $\langle \phi_i, L[\phi_j] \rangle = \omega_i^2 \delta_{ij}$, with ω_i and ϕ_i being *i*-th modal frequency and shape. However it is often criticized due to completely neglecting non-essential modes $q_l, l \neq m$, which leads to possible error predictions.

Truncation order problem: refined finite mode truncation

(1) The *NNMs* method introduces the following invariant manifolds

$$q_l = g_l(q_m, \dot{q}_m) \sim O(q_m^2, \dot{q}_m^2), \quad \dot{q}_l = h_l(q_m, \dot{q}_m) \sim O(q_m^2, \dot{q}_m^2), \quad l \neq m \tag{3}$$

to enslave the non-essential modes ($l \neq m$) to the dominant one, satisfying manifold equations (say, up to second order)

$$\frac{\partial g_l}{\partial q_m}\dot{q}_m + \frac{\partial g_l}{\partial \dot{q}_m}\left(-\omega_m^2 q_m\right) = h_l, \quad \frac{\partial h_l}{\partial q_m}\dot{q}_m + \frac{\partial h_l}{\partial \dot{q}_m}\left(-\omega_m^2 q_m\right) = -\omega_l^2 g_l + \langle \phi_l, N_2[\phi_m, \phi_m] \rangle q_m^2 \tag{4}$$

(2) The *full-basis or direct perturbation* method designs a 'ladder-type' expansion scheme like

$$O(\varepsilon): \quad D_0^2 q_{m1} + \omega_m^2 q_{m1} = 0, \quad O(q_{l1}) \ll O(q_{m1}), \quad l \neq m \tag{5}$$

$$O(\varepsilon^2): \quad D_0^2 q_{m2} + \omega_m^2 q_{m2} = \langle \phi_m, N_2[\phi_m, \phi_m] \rangle q_{m1}^2, D_0^2 q_{l2} + \omega_l^2 q_{l2} = \langle \phi_l, N_2[\phi_m, \phi_m] \rangle q_{m1}^2, \quad l \neq m \tag{6}$$

to dynamically condense the non-essential modes ($l \neq m$) to the dominant one $q_m, \dot{q}_m$ (satisfying Eq.(6) at $O(\varepsilon^2)$)

$$q_{l2} = \frac{\langle \phi_l, N_2[\phi_m, \phi_m] \rangle}{0^2 + \omega_l^2}\left(\frac{q_m^2}{2} + \frac{\dot{q}_m^2}{2\omega_m^2}\right) + \frac{\langle \phi_l, N_2[\phi_m, \phi_m] \rangle}{-4\omega_m^2 + \omega_l^2}\left(\frac{q_m^2}{2} - \frac{\dot{q}_m^2}{2\omega_m^2}\right), \quad l \neq m \tag{7}$$

(3) The (*simplified*) *normal form* method introduces the following nonlinear near-identity transformations

$$q_l = p_l + \hat{G}_l(p_m, \dot{p}_m), \quad \dot{q}_l = \dot{p}_l + \hat{H}_l(p_m, \dot{p}_m), \quad l = 1, 2 \cdots \tag{8}$$

to reformulate the original full-basis discretized Galerkin model as

$$\begin{aligned} \ddot{p}_l + \omega_l^2 p_l &= 0 + 2\langle \phi_l, N_2[\phi_m, \phi_m] \rangle p_m \hat{G}_m \\ &+ \sum_{j=1, j\neq m}^{\infty} \langle \phi_l, N_2[\phi_j, \phi_m] + N_2[\phi_m, \phi_j] \rangle \hat{G}_j p_m + \langle \phi_l, N_3[\phi_m, \phi_m, \phi_m] \rangle p_m^3 \end{aligned} \tag{9}$$

satisfying the simplified homological equations (say, up to second order)

$$\nabla_Y \hat{M}_l [\Lambda_m] Y - [\Lambda_l] \hat{M}_l = \left[0, \langle \phi_l, N_2[\phi_m, \phi_m] \rangle p_m^2\right]^T, \hat{M}_l \triangleq \left[\hat{G}_l, \hat{H}_l\right], Y = \left[p_m, \dot{p}_m\right]^T, \nabla_Y \triangleq \left[\partial/\partial p_m, \ \partial/\partial \dot{p}_m\right] \quad (10)$$

(4) The *low-order elimination* technique [1] designs a new displacement decomposition augmented by passive patterns

$$w(x,t) = \phi_m(x) q_m(t) + \sum \Phi_{\Omega_k}(x) P_{\Omega_k}(t) \quad (11)$$

where $\left(\phi_m, q_m \sim e^{\mathrm{i}\omega_m t}\right)$ is the dominant mode and $\left(\Phi_{\Omega_k}, P_{\Omega_k} \sim e^{\mathrm{i}\Omega_k t}\right)$ is the k-th passive pattern produced by low-order quadratic source terms, with $\Omega_p \triangleq \{0, 2\omega_m\}$, satisfying

$$\sum_{\Omega_k \in \{0, 2\omega_m\}} \left(-\Omega_k^2 I + L\right) \Phi_{\Omega_k}(x) P_{\Omega_k}(t) = N_2[\phi_m, \phi_m] q_m^2, \quad B_0\left[\Phi_{\Omega_k}\right] = B_1\left[\Phi_{\Omega_k}\right] = 0 \quad (12)$$

$$P_0(t) = q_m^2/2 + \dot{q}_m^2/\left(2\omega_m^2\right), \qquad P_{2\omega_m}(t) = q_m^2/2 - \dot{q}_m^2/\left(2\omega_m^2\right) \quad (13)$$

leading to so-called low-order elimination in the reference model

$$\left(\frac{\partial^2}{\partial t^2} + L\right)\phi_m q_m = \boxed{0} + \sum_{\Omega_k \in \Omega_p} \left(N_2\left[\phi_m, \Phi_{\Omega_k}\right] + N_2\left[\Phi_{\Omega_k}, \phi_m\right]\right) q_m P_{\Omega_k} + N_3[\phi_m, \phi_m, \phi_m] q_m^3 + \cdots \quad (14)$$

Quite interestingly, the four distinctly motivated reduction methods above are equivalent to each other, and a theoretical correspondence can be established [2].

Truncation degree problem

Due to the correspondence/equivalence above, the truncation degree problem in nonlinear model reduction is developed in the low-order elimination formulation using passive patterns. We consider full/non-full truncations of degree two

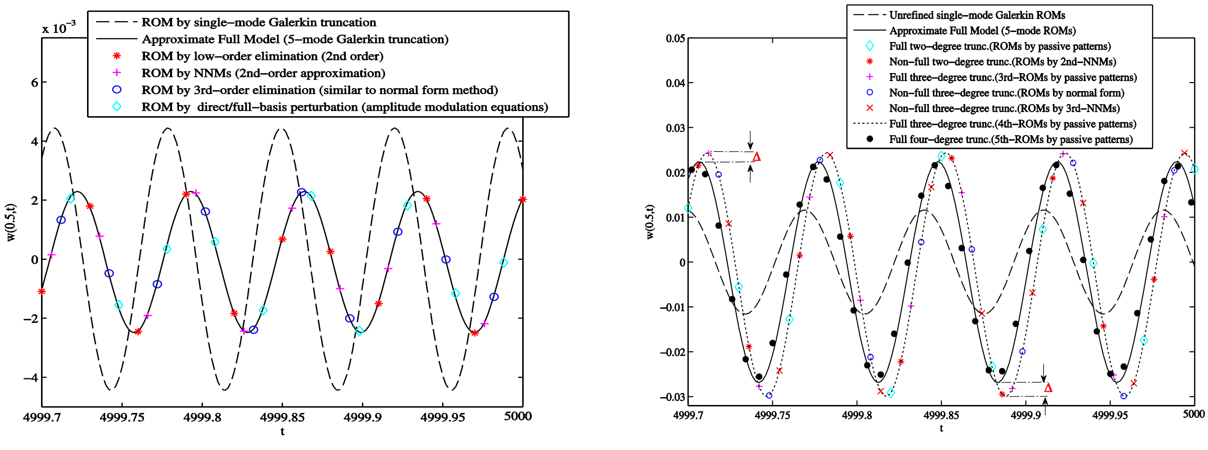
$$w(x,t) = \phi_m q_m + \Phi_0 P_0 + \Phi_{2\omega_m} P_{2\omega_m} + \cdots, \qquad w(x,t) = \phi_m q_m + \bar{\Phi}_0 P_0 + \bar{\Phi}_{2\omega_m} P_{2\omega_m} + \cdots \quad (15)$$

and also full/non-full truncations of degree three

$$\begin{aligned} w(x,t) &= \phi_m q_m + \Phi_0 P_0 + \Phi_{2\omega_m} P_{2\omega_m} + \Phi_{\omega_m} P_{\omega_m} + \Phi_{3\omega_m} P_{3\omega_m} + \cdots \\ w(x,t) &= \phi_m q_m + \Phi_0 P_0 + \Phi_{2\omega_m} P_{2\omega_m} + \bar{\Phi}_{\omega_m} P_{\omega_m} + \bar{\Phi}_{3\omega_m} P_{3\omega_m} + \cdots \\ w(x,t) &= \phi_m q_m + \bar{\Phi}_0 P_0 + \bar{\Phi}_{2\omega_m} P_{2\omega_m} + \hat{\Phi}_{\omega_m} P_{\omega_m} + \hat{\Phi}_{3\omega_m} P_{3\omega_m} + \cdots \end{aligned} \quad (16)$$

Note non-full truncation means the pattern shape functions $\bar{\Phi}_{\Omega_k}$ and $\hat{\Phi}_{\Omega_k}$ are incomplete with master components being skipped. It turns out that the NNMs method (invariant manifolds) can always be regarded as a non-full truncation (thus non-full elimination) technique, while the existing normal form method can be regarded as either a non-full three-degree truncation, or a full two-degree truncation with further third order simplification. Interestingly, for quadratic/cubic structures, all the reduction methods, either degree two or three, full or non-full, lead to equivalent third-order ROMs. Furthermore, in degenerate case, a four-degree truncation will be required [3].

The two unified perspectives on nonlinear model reduction are further illustrated in Fig.1 and the numerical results are obtained by applying the reduction methods above to a nonlinear foundation beam example.



(a) (b)

Fig.1 Unified perspectives and numerical illustrations: (a) truncation order problem [2]; (b) truncation degree problem (degenerate dynamics) [3]

References

[1] T.D Guo and G. Rega, Reduced-order modelling of nonlinear structures: A low-order elimination technique using passive patterns, submitted, 2021

[2] T.D Guo and G. Rega, On model reduction techniques of nonlinear structures: Theoretical correspondence and refined finite mode truncation, preprint, 2022

[3] T.D Guo, G. Rega, H.J, Kang, On the minimal degree of truncation for nonlinear model reduction: A low-order elimination perspective, preprint, 2022