Conservative solitons and reversibility in time delayed systems

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Summary. Time-delayed dynamical systems (TDSs) materialize in situations where distant, point-wise, nonlinear nodes exchange information that propagates at a finite speed. They are akin in their complexity to spatially extended *dissipative and diffusive* systems. However, they are considered devoid of dispersive effects, which are known to play a leading role in pattern formation and wave dynamics. It explains why the existence of nonlinear conservative TDSs remains an open problem. In this work we show how dispersion may appear naturally in a wide class of delayed systems that can lead to conservative dynamics. We exemplify our result considering a dispersive micro-cavity containing a Kerr medium coupled to a distant external mirror. At low energies and in the long delay limit, a multi-scale analysis shows the equivalence with the nonlinear Schrödinger equation.

Introduction

Delayed dynamical systems describe a large number of phenomena in nature and they exhibit a wealth of dynamical regimes such as localized structures, fronts and chimera states [1, 2, 3, 4, 5]. The presence of delayed terms is connected with the finite propagation speed of signals, hence delayed systems are widely used to model, e.g., networks, map lattices or optical systems. A fertile perspective lies in their interpretation as spatially extended *diffusive* systems which holds in the limit of long delays [6]. This correspondence enables a direct interpretation of purely temporal phenomena in terms of diffusive, dissipative spatio-temporal dynamics. It was shown recently[7] that a more general class of singularly perturbed TDSs allows to cancel this generic diffusive behavior which lead to a dispersive response and the purely imaginary eigenvalue spectrum typical of reversible systems. However, the question whether conservative solitons can be obtained in such TDSs remains open. In this contribution, we demonstrate the existence, that remained elusive so far, of nonlinear, time-reversible, conservative TDSs leading to conservative solitons.

Results

We consider a micro-cavity containing a nonlinear Kerr medium coupled to an external mirror as depicted in Fig. 1 (a). Our theoretical approach follows the method developed in [8] and the dynamical model for the slowly varying electromagnetic field envelopes in the micro-cavity E and the external cavity Y reads

$$\dot{E} = \left(-1 + i\left|E\right|^2\right) E + hY \,, \qquad (1)$$

$$Y = r_e e^{i\varphi} \left[E(t-\tau) - Y(t-\tau)\right] \,. \qquad (2)$$

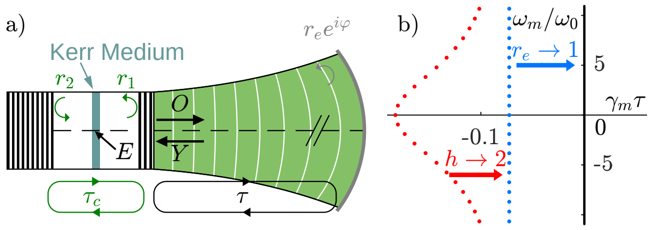

Figure 1: a) A schematic of a micro-cavity enclosed by two distributed Bragg mirrors with reflectivities $r_{1,2}$. It is coupled to a long external cavity with round-trip time τ which is closed by a mirror with reflectivity r_e. E is the sum of the forward and backward propagating fields that interfere upon the Kerr medium while Y is the field impinging upon the top mirror. b) The eigenvalue spectrum of Eqs. (3,4) for different values of h and r_e. For $h < 2$ and $r_e < 1$ (red dots) the spectrum resemble an inverted parabola. When $h \to 2$ (blue dots) it flattens as $\gamma_m \tau = \ln r_e$. For $r_e \to 1$, the spectrum converges to the imaginary axis.

We scale time to the photon lifetime in the micro-cavity and the cavity enhancement is scaled out allowing E and Y to be of the same order of magnitude leading to a simpler input-output relation: The cavity output $O = E - Y$ is the combination of the intra-cavity photons transmitted by the micro-cavity and those reflected. Finally, the field intensities are scaled by the intra-cavity enhanced Kerr saturation parameter. The coupling between the fields E and Y is given in Eq. (2). Due to the absence of the time derivative, the latter is a Delay Algebraic Equation (DAE) that takes into account all the multiple reflections in the external cavity. The field is re-injected after a round-trip τ with the attenuation factor $r_e\, e^{i\varphi}$, where r_e is the external-mirror reflectivity and the phase φ contains both the propagation phase as well as that of the external mirror. The light coupling efficiency in the cavity is given by the factor $h = (1 + |r_2|)(1 - |r_1|)/(1 - |r_1 r_2|)$, where $|r_{1,2}|$ are the upper and lower distributed Bragg mirror reflectivities (cf. Fig. 1 (a)). In particular, for a perfectly lossless bottom mirror $|r_2| = 1$ yields $h = 2$, which corresponds to the Gires-Tournois regime [9]. Second- and third-order dispersion are naturally captured by Eqs. (1,2) as was shown in [7].

The system (1,2) possesses infinitely many degrees of freedom and its eigenvalue spectrum is a countably infinite set. In the long delay limit $\tau \to \infty$ the spectrum becomes quasi-continuous [10] and the real part of the eigenvalues, obtained around the $(E, Y) = (0, 0)$ solution, $\lambda_m = \gamma_m + i\omega_m$ can be expressed as a function of its imaginary part as [7]

$$\gamma_m = \frac{1}{\tau}\left(\ln r_e + \frac{h-2}{1+\omega_m^2}\right), \tag{3}$$

$$\omega_m \tau + 2\arctan\omega_m = \frac{h-2}{1+\omega_m^2}\omega_m + 2\pi m\,, \tag{4}$$

with $m \in \mathbb{Z}$, see the red dots in Fig. 1 (b). Note that for $h = 2$ the real part γ_m does not depend on ω_m yielding a vertical, yet lossy, spectrum which is shifted from zero by $\ln r_e/\tau$ (blue dots). In what follows we consider the lossless cavity limit that corresponds to $(r_e, h) = (1, 2)$ and where λ_m converges towards *the unitary spectrum* presented in solid black in Fig. 1 (b). In itself, a unitary spectrum is rather surprising in the context of TDSs since it implies the possibility to integrate linear perturbations backward in time without any particular problem.
The link between Eqs. 4(1),(2) and the nonlinear Schrödinger (NLS) equation can be clarified by performing a multi-scale. We introduce the two time representation by defining $\sigma \in [0, \tau]$ and $\theta \in \mathbb{N}$ so that time can be expressed a $t[\sigma, \theta] = \sigma + \theta\tau$. Assuming a field with carrier frequency δ as $E(t) = A(t)\exp(i\delta t)$ one obtains the following amplitude equation

$$i\left(\partial_\theta + \upsilon\partial_\sigma\right)A + \tilde{\varphi}A - \frac{\beta_2}{2}\partial_\sigma^2 A - i\frac{\beta_3}{6}\partial_\sigma^3 A + \gamma\left|A\right|^2 A = 0, \tag{5}$$

that is, the NLS equation with third order dispersion. Changing the carrier frequency allows to modify the sign and even cancel the second order dispersion coefficient β_2 which corresponds to the transition from anomalous to normal dispersion while third order dispersion β_3 can also cancel for some particular value of the carrier frequency $\delta_\pm = \pm 1/\sqrt{3}$. In this case Eq. (5) is equivalent to the classical NLS equation up to fourth order. From this equivalence, we observed that in the anomalous dispersion regime the Eqs. (1),(2) allows for the exisistence of bright hyperbolic secant solitons while we also observed dark solitons for normal dispersion. Our results were confirmed using the continuation package ddebiftool [11]. Finally, one can rewrite the Eqs. (1),(2) as a Neutral Differential Delayed Equation (NDDE)

$$E + \dot{E} - i\left|E\right|^2 E \;\; = \;\; \left(E_\tau - \dot{E}_\tau + i\left|E_\tau\right|^2 E_\tau\right)e^{i\varphi}. \tag{6}$$

The Eq. (6) is a so-called bilateral NDDE which preserves its type under time inversion. This property, as well as the presence of a purely imaginary nonlinearity that corresponds to the Kerr effect allows to demonstrate the reversibility in time of the dynamics under the parity-conjugation symmetry.

Conclusions

We discussed how second and third order dispersion can be implemented in delayed dynamical systems using delay algebraic equations and how this particular form of dynamical systems appears naturally as a boundary conditions on a partially reflecting micro-cavity. Using a realistic photonic example, we have demonstrated that nonlinear reversible TDSs exist and that they can host conservative solitons in the long delay limit, thereby bridging the gap with the results known for dissipative TDSs. The essential structure consists of a bilateral NDDE with an imaginary cubic nonlinearity which preserves the solution smoothness upon forward and backward propagation and may generate a unitary spectrum. The normal form identifies with the NLS equation, thereby allowing for bright and dark solitons. We believe that bilateral NDDEs open an avenue for the potential realization of conservative nonlinear dynamics in TDSs, such as, e.g., the Fermi-Pasta-Ulam-Tsingou [12] recurrence or the observation of the Korteweg-de-Vries solitons.

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