Feedback control of propagating bubbles in Hele-Shaw channels

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Summary. We explore the capabilities of feedback control to stabilise and manipulate propagating bubbles in the confined geometry of a rectangular Hele-Shaw channel. Several steadily-propagating solution branches exist in this system, but only one is linearly stable. Subsequent branches featuring increasingly deformed bubble shapes and increasing numbers of unstable eigenmodes. Our aim is to use feedback control and control-based continuation to detect and stabilise at least the first of these unstable branches. This system is an appealing prototype for control: the low Reynolds number and strongly confined geometry means the system state is essentially encapsulated in the interface shape as viewed from above, recent experimental realisations of this system are in good agreement with depth-averaged models, and the system responds well to actuation via fluid injection, but nonetheless practical implementation of feedback control presents significant challenges. Here we use a depth-averaged model to explore how control would work in this system, including the design of a suitable feedback gain, the impact of control on the bifurcation structure, the complexities of controlling a propagating bubble moving past a fixed array of injection points, and how our idealised simulations relates to experimental reality.

Introduction

The study of nonlinear dynamics in soft matter systems often leads to a myriad of steady states. Stable states can be studied via both experimental and computational methods but experimental investigations at best transiently observe unstable states. Control based continuation (CBC) is a methodology that applies feedback control to the physical system using a target state as the control parameter. By choosing an unstable state as the target state and carefully choosing a feedback strategy it is possible to detect and stabilize the unstable solution without otherwise modifying it. This methodology enables tracking of bifurcation structures as well as experimental investigations of unstable behaviour [1, 2, 3]. Our aim is to use the CBC to investigate free-surface problems in fluid dynamics, both to understand the rich range of dynamical behaviour and to widen the range of possible applications of CBC in the future.

We study the propagation of an air bubble through a fluid filled rectangular Hele-Shaw channel at low Reynolds number. This system is a classical problem in fluid dynamics with known rich dynamics, which we have previously explored both via a depth-averaged model and also via laboratory experiments [4]. For steady propagation, the bubble system supports one stable solution (figure 1(a)) and an infinite number of multi-tipped unstable solutions that assume an increasing number of interface tips and unstable eigenvalues (figure 1(b,c)). Our aim is to develop and test an experimentally viable protocol for the stabilization of unstable solutions, to detect and stabilise at least the first unstable solution branch.

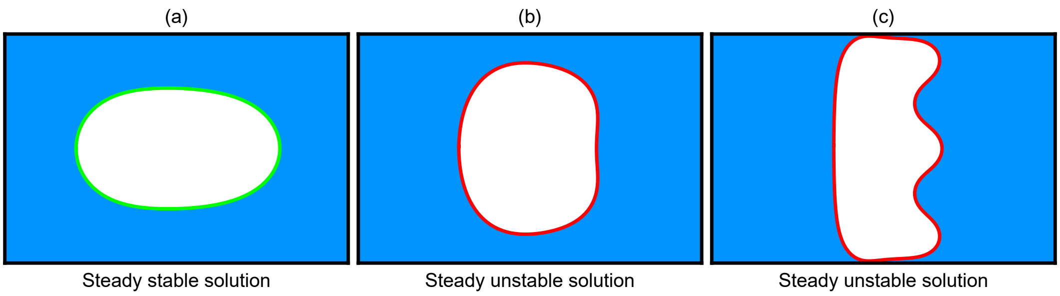

Figure 1: Three steady modes for a bubble propagating from left to right in a Hele-Shaw channel, calculated using our model.

Model

We take advantage of the shallowness of Hele-Shaw channels by using a depth-averaged approximation of the system, thus avoiding the computational complexity of the full 3D problem for bubble propagation. The problem domain is then two-dimensional, with the interface shape corresponding to the top view seen in experiments. The numerical simulations are implemented in C++ using the open-source object-oriented multi-physics finite element library oomph-lib (www.oomph-lib.org).

Our model is inevitably an approximation of the full experimental system and does not capture all details of the bifurcation structure in the experiments. Nonetheless, the model is in good qualitative agreement with experiments and in quantitative agreement in appropriate regimes. As a result, we have reason to believe that control strategies developed using this model would be effective, though not optimal, if applied in experiments.

Control via moving or fixed actuators

In order to actuate control, we inject fluid from specific points at the side walls with time-dependent amplitude $\mathbf{A}(t)$. The system can be generically represented as $\mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{A}) = \mathbf{0}$ where $\mathbf{x}(t)$ is a vector that describes the system state (the current interface shape) and $\mathbf{A}(t)$ describes the injection amplitude. The simplest case to analyse is when actuator position is fixed within the frame of the bubble, so actuator action is steady as perceived by the bubble. For simplicity we use a linear feedback control strategy $\mathbf{A}(\mathbf{t}) = -\mathbf{K}(\mathbf{x}(\mathbf{t}) - \mathbf{x_t})$ towards a target state $\mathbf{x}_T$. Various standard algorithms can be employed to choose a feedback gain $\mathbf{K}$ that (linearly) stabilizes the resulting steady solution.

We test our control strategies in nonlinear time-dependent simulations of the depth-averaged model. Choosing the unstable solution of the uncontrolled system (figure 1(b)) as the target state $\mathbf{x}_T$, the stable solution (figure 1(a)) as the initial condition, and using a pair of actuators placed a distance $d = 1$ (half the channel width) in front of the bubble centroid, we can successfully control the system towards the target solution, see figure 2(a). The transition between states is rapid and the bubble moves only a short distance along the channel.

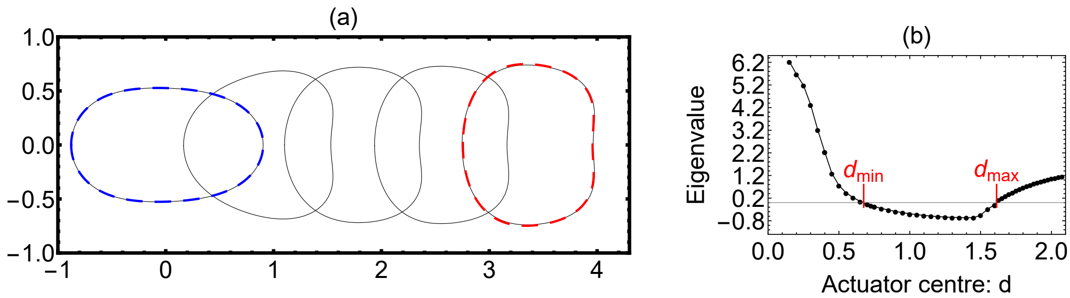

Figure 2: (a) Time evolution of a bubble, presented in the lab frame, starting with the stable solution as initial condition and evolving towards the unstable target solution. The actuator distance d is half of the channel width. The first and last interfaces are superposed with the stable and unstable solutions, in blue and red dashed lines respectively. (b) Eigenvalue plotted as a function of the actuator centre d. The steady state is successful stabilized for values of d between d_{min} and d_{max}.

A more realistic setup actuator setup is via an array of injection points fixed in the lab frame. In the frame of the bubble, this introduces time dependence in the control problem as the distance d between actuator and bubble depends on time. However, each fixed $\mathbf{K}$ has a window of d where the control is able to stabilize the equilibrium state (figure 2(b)). Hence by choosing actuator spacing within this range and switching the active actuators depending on the position of the bubble we are again able to stabilize bubbles in our simulations. Importantly for CBC, this piecewise constant $\mathbf{K}$ means that a low-dimensional parameterization of the target state is still sufficient for feedback control.

Outlook for experiments

Our analysis and simulations of control based on a depth-averaged model suggest that an unstable state can be successfully controlled via injection of fluid chosen in real time based on top view observations of the bubble. Compared to simulations, experiments present several new challenges, including noise and delay. CBC can in principle address the inevitable differences between experiment and model, as the model would be used to devise an effective control strategy which need not be optimal. It remains to be seen how closely the bifurcation structure, solution shapes and stability properties for unstable states in experiments corresponds to those predicted by the simplified model used here. We are currently developing an experimental setup to test and implement these control strategies in practice and will present experimental results for either static or moving bubbles if possible.

References

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