Analysis of parametric instabilities of two-oscillator bi-linear model through the resonant order reduction in the vicinity of similar nonlinear normal modes

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Summary. Present study is devoted to analysis of fundamental problem of parametric instability of damped-driven bi-linear two-oscillator model, given to the low-amplitude parametric forcing. Assuming the resonant excitation in the vicinity of similar NNMs, we derive the reduced order model. Further, applying the method of isolated resonance we analyze asymptotically the most significant family of sub-harmonic (m:1) resonance regions which reveal their peculiar properties.

Introduction

Response of bi-linear oscillatory systems subject to various types of forcing has become a subject of immense theoretical and experimental research. These models are broadly applied for mathematical modeling of the response of various engineering problems such as mooring towers [1], interlocking structures [2], suspension bridges [3], beams with breathing cracks [4] and many more. Numerous analytical and numerical works have been devoted to the analysis of externally forced, single bi-linear oscillator. Shaw and Holmes [5] were the first to develop the semi-analytical methods for the analysis of stability and bifurcation structure of periodic orbits emerging in the harmonically forced piece-wise linear oscillator (PWLO). In the same year, a thorough computational study of sub-harmonic orbits, bifurcations of periodic solutions and chaotic motion of harmonically forced piece-wise linear oscillator has been reported by Thomson et al. [6]. Some initial experimental work by Shaw [8] has shown the superharmonic and sub-harmonic steady state regimes in the experimental setup, mimicking the damped-forced response of piece-wise linear oscillator. We bring here some most fundamental theoretical works which considered the parametric instability phenomena arising in the various damped-driven piece-wise linear setups [9, 10, 11].

Model

In the present study we consider the damped-driven system of two coupled identical, bi-linear oscillators which assume the parametric excitation on the first oscillator. The non-dimensional equations of motion of the system under consideration read,

$$\xi_1^{''} + \epsilon\lambda\xi_1^{'} + [1+\alpha\mathcal{H}(-\xi_1)]\,\xi_1 + \beta\,(\xi_1-\xi_2) + \epsilon P\cos(\Omega t)\xi_1 = 0\ , \tag{1a}$$

$$\xi_2^{''} + \epsilon\lambda\xi_1^{'} + [1+\alpha\mathcal{H}(\xi_2)]\,\xi_2 + \beta\,(\xi_2-\xi_1) = 0\ , \tag{1b}$$

. Here ϵ is a formal, small non-dimensional system parameter. Where λ, P, α, β stand for the damping, forcing, bi-linear nonlinearity and coupling parameters. The formal small system parameter $0 < \epsilon << 1$ is introduced for scaling the magnitude of damping and forcing terms. This system can model the low amplitude vibrations of two linearly coupled, damped pendulums where one out of the two pendulums, performs the prescribed vertical oscillatory motion on its point of suspension.

It can be easily seen that OP NNN belongs to the special family of NNMs of similar type. Further setting the OP NNM to be the resonating mode, we define the two auxiliary coordinates $\xi_1 - \xi_2 = \eta$ and $\xi_1 + \xi_2 = \zeta$. It is reasonable to assume that in the resonant motion of the system being parametrically excited in the resonant region of OP NNM, the newly defined η variable, dominates over ζ. Applying some rather simple algebraic manipulation and using the same resonant assumption, one readily arrives at the following, reduced order model (ROM) i.e. the damped-driven, bi-linear oscillator which recovers the entire resonance structure of resonating OP NNM.

$$\eta^{''} + \epsilon\lambda\eta^{'} + (1+\alpha\mathcal{H}(-\eta)+2\beta)\,\eta + \epsilon\frac{P}{2}\cos((m\Omega_N+\epsilon\sigma)t)\eta = 0\ . \tag{2}$$

Results

We rewrite Eq. (2) in the phase space form, as follows:

$$q^{'} = p, \quad p^{'} = -\epsilon\lambda p - (1+\alpha\mathcal{H}(-q)+2\beta)\,q + \epsilon\frac{P}{2}\cos((m\Omega_N+\epsilon\sigma)t)q\ , \tag{3}$$

It is worthwhile noting that the asymptotic analysis applied in the present work is valid up to $\mathcal{O}(\epsilon)$ order. Therefore, the terms of the higher asymptotic order are omitted in the following part of analysis. To derive the asymptotic approximation for transition curves corresponding to the special family of (m:1) resonant tongues, we introduce the action-angle variables [11, 12]

$$I(E) = \frac{1}{2\pi}\oint p(q,E)dq\ , \quad \text{and} \quad \Theta = \frac{\partial}{\partial I}\int_0^q p(q,I)dq\ . \tag{4}$$

and using the method of isolated resonance derive the averaged flow in the neighborhood of (m:1) resonance tongues.Further analysis of the averaged flow, reveals the closed form asymptotic approximations for the transition curves for different resonance conditions (see e.g. [12]). Please note that $\epsilon\sigma$ stands for the small detuning parameter, while the excitation frequency in the vicinity of m:1 resonant neighborhood is taken as $\Omega = m\Omega + \epsilon\sigma$. Here Ω_N is a natural frequency of the out-of-phase NNM. In Figure. 1 we present the stability charts for the three resonant cases. It can be readily seen that the analytical approximation of transition curves (shown by black solid lines) are in very good agreement with the results of numerical simulations of the full model (Eqs. (1a) and (1b)). In Figure.2 (left panel) we present the evolution of transition curve w.r.t damping. In the limiting case of zero damping transition curves become straight lines whose slopes change w.r.t the system parameter α. As is shown in 2 (right panel) the width of the instability regions, varies w.r.t the change in the parameter of non-linearity, such that the well-known, fundamental, 2:1 parametric resonance tongue can become narrower than the 1:1 tongue.

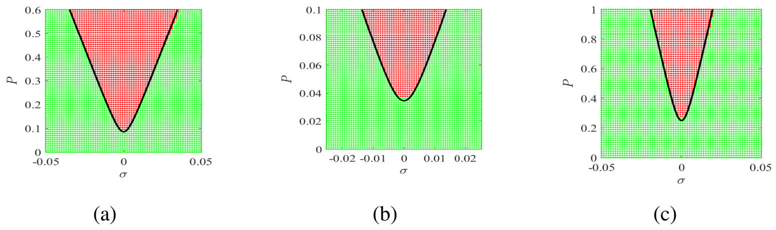

Figure 1: Stability chart for resonance conditions (a) $1:1$, (b) $2:1$, and (c) $3:1$. Here green and red colours represent stable and unstable regions, respectively obtained numerically from exact model in Eqs. (1a) and (1b) and black solid lines represent instability boundaries obtained using A-A method. Parameters are $\alpha = 2$, $\beta = 0.5$, $\lambda = 0.005$ and $\epsilon = 0.1$.

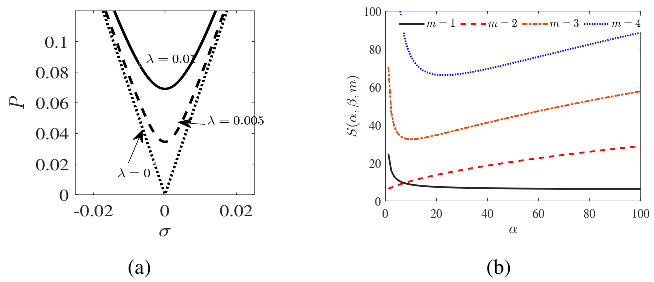

Figure 2: (a) Variations of the instability tongues with damping coefficient λ for resonance condition $2:1$. (b) Variations of the slope corresponding to each resonance condition with α.

Concluding Remarks

Present study has concerned the resonant parametric excitation of the symmetric system of two coupled bi-linear oscillators. To derive the relatively simple analytical approximation to transition curves, we derive the reduced order model which mimics the response of resonantly excited two-oscillator model in the vicinity of out-of-phase NNM. Boundaries of the transition regions emerging in the original, parametrically forced system are approximated asymptotically. Transition curves emerging from the analysis of effective bi-linear, parametric oscillator are in very good agreement with numerical results of the original system. In this talk, we will also present the alternative analytical treatment for the case of resonant system reduction in the vicinity of NNMs of the general type. This part of the talk is currently a work in progress.

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