A General Bayesian Nonlinear Estimation Method Using Resampled Smooth Particle Hydrodynamics Solutions of the Fokker-Planck Equation

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Summary. The state estimation problem for noisy nonlinear systems remains a difficult problem, particularly as the dimension of the state space grows large. This presentation will consist of a brief introduction to the problem as a diffusion process and its solution employing smooth particle hydrodynamics (SPH) to advance the estimator through the state space in time. Performance comparisons between the current algorithm, the particle filter, and the extended Kalman filter for Duffing systems of two and four dimensions will be presented.

Introduction

The effectiveness of a nonlinear estimator in a noisy environment depends on many factors, one being the accuracy with which it can predict the state dynamics of the underlying dynamical system between measurements. It is well known that a memoryless nonlinear dynamical system driven by additive and/or multiplicative Gaussian white noise can be represented by a system of D nonlinear stochastic differential equations of the Ito form, where D is the number of system states. The Bayesian optimal prior can be obtained by solving the corresponding Fokker-Planck Equation (FPE), governing the evolution of the transition probability density function of the system response over the state space [1]. The FPE is a degenerate, linear, elliptic-parabolic partial differential equation having D spatial dimensions, plus time, on an infinite spatial domain, for $t \geq 0$. To date, for the nonstationary (transient) problem, analytical solutions exist only for the scalar case, $D = 1$, the exception being the linear system subject to additive noise for which the analytical solution can be found for arbitrary D. Thus, although computational solutions of the nonstationary FPE for nonlinear systems of dimension $D = 3$ have been tractable since the mid-1980s, solution remains problematic for realistic systems with $D > 3$ due to scaling issues caused by the well-known "curse of dimensionality," and it remains extraordinarily difficult and costly to achieve accurate solutions to higher dimensional problems over the entire state space [2,3].

This presentation is a summary of our recent work addressing a general nonlinear filter based on solving the nonstationary FPE in $\mathbf{R}^D$ using Smooth Particle Hydrodynamics (SPH) at lower resolution which, for the limited number of four-dimensional systems studied, appears to result in reasonably accurate state estimation results. The filter is enabled by an efficient heuristic resampling scheme of the SPH solution, also briefly discussed. The resulting FPE-SPH filter appears able to replicate the accuracy of both the well-known, simulation-based Particle Filter (PF) and linearization-based Extended Kalman Filter (EKF) for lower dimensional systems, while being more robust than the EKF, at least for the several higher-dimensional systems examined [4].

In the limited time available, a short exposition of the underlying theory will be given, followed by a comparison of results obtained for two-dimensional and four-dimensional Duffing oscillators.

Background

Consider a system of stochastic differential equations of the form

$$d\mathbf{x} = \mathbf{f}(\mathbf{x},t)dt + \mathbf{g}(\mathbf{x},t)d\mathbf{W}(t) \tag{1}$$

$$\mathbf{y} = \mathbf{h}(\mathbf{x},t) + \mathbf{v}(t) \tag{2}$$

where $\mathbf{x},\mathbf{f} \in \mathbb{R}^D$, $\mathbf{g} \in \mathbb{R}^{D\times D_N}$, and $\mathbf{w} \in \mathbb{R}^{D_N}$, subjected to zero mean Gaussian white noise that defines a Wiener process $d\mathbf{W}(t) = \sqrt{t}\mathbf{w}$, $E[\mathbf{w}(t)] = 0$, $E[\mathbf{w}(t)\mathbf{w}^{\mathrm{T}}(t+\tau)] = \mathbf{Q}\delta(\tau)$. Also, $\mathbf{y}$ is a state measurement process, where $\mathbf{y},\mathbf{h} \in \mathbb{R}^{D_o}$ and $\mathbf{v} \in \mathbb{R}^{D_o}$ is a nonzero mean Gaussian white noise, and $E[\mathbf{v}(t)] = 0$, $E[\mathbf{v}(t)\mathbf{v}^{\mathrm{T}}(t+\tau)] = \mathbf{R}\delta(\tau)$. There exists a corresponding Fokker-Planck-Kolmogorov equation which defines the evolution in time of the transition probability density function of the system states, which takes the form

$$\frac{dp}{dt} = -\sum_{i=1}^{D}\frac{\partial}{\partial x_i}(D_i^{(1)}p) + \frac{1}{2}\sum_{i=1}^{D}\sum_{j=1}^{D}(D_{ij}^{(2)}p) \tag{3}$$

subject to

$$p(\mathbf{x},0\,|\,\mathbf{x}_0) = \prod_{i=1}^{D}\delta(x_i - x_{i0}) \tag{4}$$

where $D_i^{(1)}$ and $D_{ij}^{(2)}$ are the deviate moments derived from eq. (1).

Results and Conclusions

For systems with simpler dynamics, the FPE-SPH, PF and EKF filters perform very similarly when properly parameterized. Differences emerge for systems such as the Duffing where the double-well potential of the PDF can result in rapid changes in an individual transient trajectory. In an estimation context sufficient process noise, measurement noise, and/or time between update steps can result in extreme EKF divergence. The PF and FPE-SPH filters on the other hand are robust against these sudden switches, which can be seen in the single run history below in Figure 1 and 25 run NEES in Figure 2.

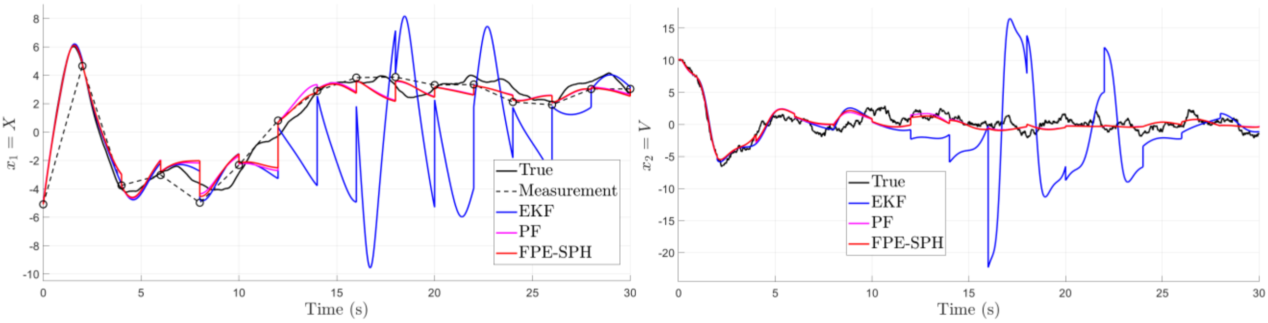

Figure 1: Example 2D Duffing estimated state histories (position, velocity) demonstrating EKF divergence.

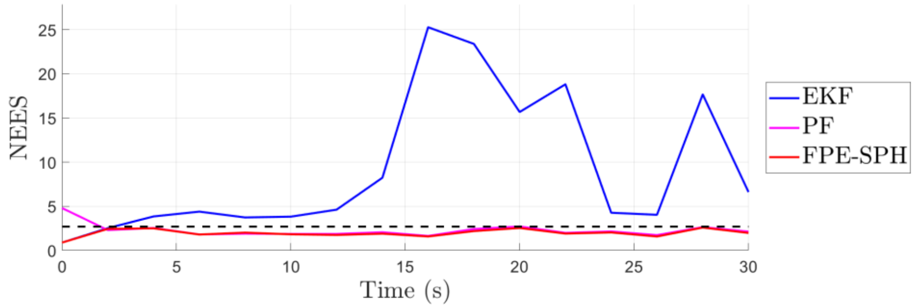

Figure 2: 2D Duffing NEES (25 Monte-Carlo runs, different initial conditions, common seeds between filters)

Robustness of the FPE-SPH filter extends up to four-state systems where one of the oscillators possesses a Duffing term. The system parameters simulated are not severe enough to result in EKF divergence as with the two state Duffing, but the results confirm that the FPE-SPH Filter is an accurate estimator in higher dimensional systems. The RMS error is shown in Figure 3.

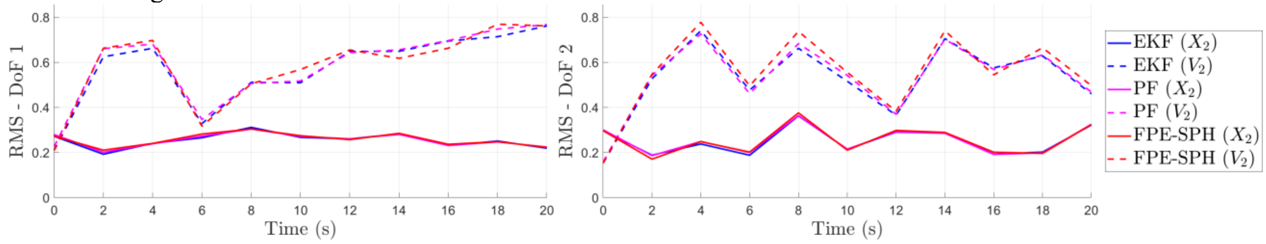

Figure 3: RMS Errors for 4D Nonlinear System, separated by degree of freedom (10 runs)

The FPE-SPH filter can accomplish this with only 5,000 particles using conservative runtime acceleration parameters compared to the PF's 100,000. Improvements to the underlying algorithm to better handle higher dimensional behavior and more aggressive parameterization to further prioritize run time might allow for further scaling to tackle systems with more than four states. Please refer to the references for information about the algorithms employed.

References

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