Non-smooth inverted pendulum swing-up control optimization using a novel, Fourier series based numerical method

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Summary. Swing-up control of an inverted pendulum is one of the classical tasks in dynamics and control theory. Optimization of the pendulum's trajectory and the corresponding control function is not trivial, particularly when the controlled object is non-smooth (for example, due to presence of dry friction). This short paper contains a concise description of a novel, Fourier series based numerical method of control optimization, which is able to successfully solve such problem. It is expected that this algorithm will enable progress in control optimization of systems, for which typical approaches cannot be utilized.

Introduction

Optimal control is the one that minimizes the cost of performing a desired action [1]. In the case of systems described by a set of ordinary differential equations (ODEs), the necessary conditions for control optimality have been described by Pontryagin [2] in terms of variational calculus. Another classical approach to solving optimal control problems is the dynamic programming method, introduced by Bellman et al. [3]. However, the former cannot be utilized in non-smooth and discontinuous systems, whereas the latter requires significant computing power even in simple cases [1]. Therefore, there is a need for a simple method which solves optimal control problems in non-smooth systems. In this paper authors present a novel, Fourier series based numerical algorithm [4] applied in optimization of the swing-up control of an inverted pendulum [5] with a dry friction discontinuity.

System description

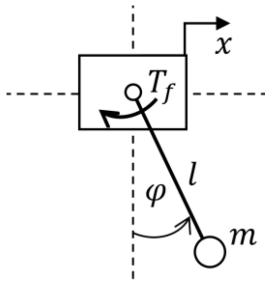

Figure 1: Scheme of the inverted pendulum system

The inverted pendulum system [5], whose scheme is presented in Fig. 1, consists of the controlled cart, able to move along the x axis, with a mathematical pendulum of the mass m and the length l attached to it. Dry and viscous friction torque in the pendulum's bearing is taken into account, i.e. $T_f = c_v\dot{\varphi} + c_d \operatorname{sgn} \dot{\varphi}$. The swing-up control means moving the cart in such a manner that the pendulum stands up from the initial state $\varphi = \dot{\varphi} = x = \dot{x} = 0$ (pendulum hanging vertically downwards) to the final one $\varphi = \pi, \dot{\varphi} = x = \dot{x} = 0$ (pendulum standing vertically upwards). Defining the dimensionless time $\tau = \omega t$ with $\omega = \sqrt{\frac{g}{l}}$, transforming the derivatives $\dot{\varphi} = \frac{d\varphi}{d\tau}\frac{d\tau}{dt} = \varphi'\omega,\ \ddot{\varphi} = \omega^2\varphi'',\ \ddot{x} = \omega^2 x''$ and introducing dimensionless parameters $z = \frac{x}{l}, \zeta = \frac{c_v}{2m\omega l^2}, \mu = \frac{c_d}{m\omega^2 l^2}$ yield the following dimensionless form of the model.

$$\varphi'' = -\sin\varphi - z''\cos\varphi - 2\zeta\varphi' - \mu \operatorname{sgn}\varphi'$$

The quantity $u = z' = \frac{\dot{x}}{\omega l}$, a dimensionless counterpart of the cart's velocity, is the controlled variable in the system. Obviously, velocity and acceleration of any physical drive are bounded. Moreover, the space in which the cart moves may be restricted. Therefore, the following constraints are assumed: $-z_{max} \le z \le z_{max}$, $-v_{max} \le z' \le v_{max}$, $-a_{max} \le z'' \le a_{max}$, where $z_{max} > 0,\ v_{max} > 0,\ a_{max} > 0$.

Optimization method

Assume that the goal is to minimize two factors in parallel, motion time T and drive usage - the integral of u'^2. Therefore, the cost functional can be defined as follows: $J = \int_0^T (\lambda + u'^2)d\tau$, where λ is a positive, real parameter. The proposed optimization method assumes that the control function $u(\tau)$ to be optimized is described using a finite number of harmonics of the Fourier series.

$$u(\tau) = \frac{a_0}{2} + \sum_{i=1}^{K}[a_i\cos(i\omega\tau) + b_i\sin(i\omega\tau)]$$

Bearing in mind that $u(\tau) = z'$ and taking into account the boundary conditions $z(0) = z'(0) = z(T) = z'(T) = 0$, it can be easily shown that $a_0 = 0$, $a_K = -\sum_{i=1}^{K-1} a_i$, $b_K = -\sum_{i=1}^{K-1} \frac{b_i}{i}$. Consequently, the control function $u(\tau)$ depends on $(2K-2)$ independent parameters, which can be collected in a vector $\boldsymbol{H} = [a_1, b_1, a_2, b_2, \dots, a_{K-1}, b_{K-1}] \in \mathbb{R}^{2(K-1)}$. Taking a large enough value K enables to approximate $u(\tau)$ with arbitrarily high accuracy. Now an important fact must be noticed: the *direction* of the vector $\boldsymbol{H}$ in $\mathbb{R}^{2(K-1)}$ is responsible for the *shape* of the function $u(\tau)$, i.e. signs of its derivatives of all degrees, locations of local extrema and inflection points etc., whereas the *length* of $\boldsymbol{H}$ determines the *span*, i.e. the set of values and the global extrema of $u(\tau)$ [4]. Therefore, the *shape* of the function $u(\tau)$ for a fixed ω can be uniquely defined by a *direction* $\mathbb{R}^{2K-2}$, which in turn can be described by a point on a unit hypersphere immersed in $\mathbb{R}^{2K-2}$, i.e. on S^{2K-3}, a $(2K-3)$ −dimensional one. Location of any point on such hypersphere depends on $2K-3$ angular coordinates, which uniquely define the *shape* of $u(\tau)$ [4]. Assume that this *shape* is already known. Now, as the continuous function $z' = u(\tau)$ is constrained by $-z_{max} \le z \le z_{max}$, $-v_{max} \le z' \le v_{max}$, $-a_{max} \le z'' \le a_{max}$, there exists the smallest number $Q > 0$ such that at least one of the constraints is violated by $q * u(\tau)$ if $q > Q$. Consequently, as the shape of $u(\tau)$ is fixed, then $p * Q * u(\tau)$ with a parameter $p \in (0, 1]$ define all the admissible functions $u(\tau)$ of the shape defined by a point on S^{2K-3}. Summing up, K-harmonics approximation of any Dirichlet control function $f: [0, T] \subset \mathbb{R} \to [-v_{max}, v_{max}] \subset \mathbb{R}$ can be described using $(2K-3)$ angular coordinates on S^{2K-3} and the parameters ω, p. Global optimization of all these parameters, for example using Differential Evolution method [6], leads to optimization of the control function $u(\tau)$ itself. When the proposed approach is utilized, smoothness of the controlled system is not required as long as the required trajectories exist and are unique.

Results and conclusions

The shape of the function $u(\tau)$ was optimized for $K \in \{2, 3\}$ parametrized in terms of spherical coordinates on S^{2K-3} along with the parameters $\omega \in [0.1, 10], p \in (0, 1]$ using the Differential Evolution method [6]. The results are presented in the graphs below.

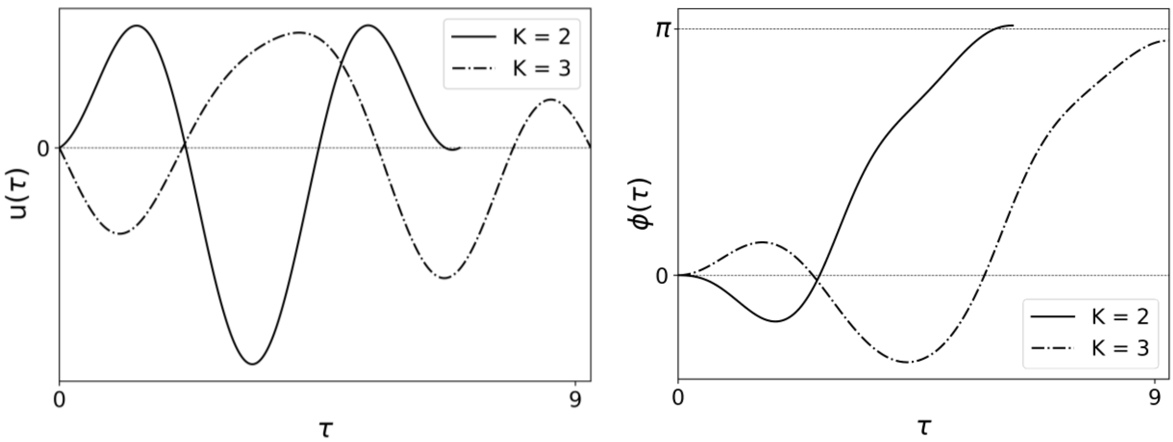

Figure 2: Results of optimizing the control function $u(\tau)$ with respect to the cost functional J.

The optimized values of the cost functional $J = \int_0^T (\lambda + u'^2) d\tau$ were approximately equal 13.93 (optimization with 2 harmonics, i.e. $K = 2$) and 12.62 (optimization with 3 harmonics, i.e. $K = 3$). Therefore, it can be noticed that increasing number of harmonics in the optimized function $u(\tau)$ from $K = 2$ to $K = 3$ increases accuracy of control optimization. It is expected that larger values of K lead to better approximations of the optimal control. As it can be noticed, the method works successfully in the discontinuous system. Authors hope that this algorithm will enable progress in control optimization of systems, for which typical approaches cannot be utilized.

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