Methods of perturbation theory and their applications in nonlinear fracture mechanics and continuum damage mechanics

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Summary. Perturbation theory techniques and their applications in nonlinear fracture mechanics (NFM) are discussed. This study summarizes an overview of the state of the art on the asymptotic description of fracture of nonlinear and damaged materials. The asymptotic stress, strain and damage fields near the crack tip for power-law materials and the influence of the damage accumulation processes on the stress-strain state in the vicinity of the crack tip are analyzed. The paper gives a detailed review of the fundamental results obtained in NFM by means of asymptotic methods and perturbation theory approaches. The main attention is paid to power-law materials and asymptotic stress and strain fields in the vicinity of the crack in both non-damaged materials and damaged materials under mixed mode loadings. The paper analyses the development of the asymptotic elastic-plastic crack-tip fields derived by Hutchinson, Rice and Rosengren as a singular dominant term of the asymptotic expansion for the stress and strain fields in a power-law hardening material and shows the current state of the asymptotic methods and their applications in NFM and continuum damage mechanics.

Introduction

Asymptotic methods have several essential advantages: universality, the analytical form of the solutions obtained and simplicity of further qualitative analysis. The asymptotic methods and perturbation theory are promising and effective approach of the derivation of approximate or even closed form solutions. Nowadays the perturbation theory techniques are applied to a wide variety of static and dynamic solid mechanics problems. Asymptotic methods have also been used with success in various nonlinear problems of fracture mechanics. The asymptotic analysis of the stress and strain distributions in the vicinity of the wedge-shaped domain is one of the most fundamental problems both in linear fracture mechanics and NFM. Thus, in linear fracture mechanics very important results have been obtained by use of the methods of asymptotic analysis. An excellent review of the most considerable contributions in this field is provided by A. Carpinteri and M. Paggi in their work [1]. The present study is aimed at analysis of the results recently obtained in NFM and continuum damage mechanics for power-law constitutive equations and the power-law damage evolution equation by means of asymptotic methods and perturbation theory.

Asymptotic solutions to problems of NFM

Crack-tip stress and strain singularities for pure power law material response $\varepsilon/\varepsilon_0 = \alpha\left(\sigma/\sigma_0\right)^n$, where α is a material constant, σ_0 is the reference yield strength, n is the strain hardening exponent, $\varepsilon_0 = \sigma_0/E$ is the reference yield strain, are investigated in [2-4]. The crack tip fields can be derived in the separable form [2-4]

$$\sigma_{ij}(r,\theta) = \sigma_0\left(\frac{J}{k_n r}\right)^{1/(n+1)}\tilde{\sigma}_{ij}(\theta,n),\ \varepsilon_{ij}(r,\theta) = \alpha\varepsilon_0\left(\frac{J}{k_n r}\right)^{n/(n+1)}\tilde{\varepsilon}_{ij}(\theta,n),\ u_i(r,\theta) = \alpha\varepsilon_0\left(\frac{J}{k_n r}\right)^{n/(n+1)} r^{1/(n+1)}\tilde{u}_i(\theta,n), \quad (1)$$

where J is the path-independent integral, I_n is the dimensionless J-integral (an integration constant depending on n), $k_n = \alpha\sigma_0\varepsilon_0 I_n$. The asymptotic fields (1) are referred to as the Hutchinson-Rice-Rosengren (HRR) fields in the vicinity of the crack tip in power-law materials. The asymptotic solution (1) was sought in the separable form $\chi\left(r,\theta\right) = r^{\lambda+1}f(\theta)$. where $\chi\left(r,\theta\right)$ is the Airy stress function: $\sigma_{\theta\theta} = \chi_{,rr},\ \sigma_{rr} = \Delta\chi - \sigma_{\theta\theta},\ \sigma_{r\theta} = -r^{-1}\left(r^{-1}\chi_{,\theta}\right)_{,r}$. The resulting nonlinear ordinary differential equation following from the compatibility equation is homogeneous in $f(\theta)$:

$$\begin{aligned}
&f_e^2 f^{IV}\left\{(n-1)\left[\left(1-\lambda^2\right)f+f''\right]^2+f_e^2\right\}+(n-1)(n-3)\left\{\left[\left(1-\lambda^2\right)f+f''\right]\left[\left(1-\lambda^2\right)f'+f'''\right]+4\lambda^2 f'f''\right\}^2\left[\left(1-\lambda^2\right)f+f''\right]+\\
&+(n-1)f_e^2\left\{\left[\left(1-\lambda^2\right)f'+f'''\right]^2+\left[\left(1-\lambda^2\right)f+f''\right]\left(1-\lambda^2\right)f''+4\lambda^2\left(f''^2+f'f'''\right)\right\}\left[\left(1-\lambda^2\right)f+f''\right]+2(n-1)f_e^2\times\\
&\times\left\{\left[\left(1-\lambda^2\right)f+f''\right]\left[\left(1-\lambda^2\right)f'+f'''\right]+4\lambda^2 f'f''\right\}\left[\left(1-\lambda^2\right)f'+f'''\right]-C_2 f_e^4\left[\left(1-\lambda^2\right)f+f''\right]+f_e^4\left(1-\lambda^2\right)f''+\\
&+C_1(n-1)f_e^2\left\{\left[\left(1-\lambda^2\right)f+f''\right]\left[\left(1-\lambda^2\right)f'+f'''\right]+4\lambda^2 f'f''\right\}f'+C_1 f_e^4 f''=0,\quad f_e^2=\left[\left(1-\lambda^2\right)f+f''\right]^2+4\lambda^2 f'^2
\end{aligned} \quad (2)$$

where the following notations are adopted: $C_1 = 4\lambda\left[(\lambda-1)n+1\right],\quad C_2 = (\lambda-1)n\left[(\lambda-1)n+2\right]$.

The fourth order nonlinear differential equation (2) with traction-free boundary conditions

$$f\left(\theta=\pm\pi\right)=0,\quad f'\left(\theta=\pm\pi\right)=0 \quad (3)$$

defines a nonlinear eigenvalue problem in which λ is the eigenvalue and $f(\theta)$ is the corresponding eigenfunction. Thus, the eigenfunction expansion method results in the nonlinear eigenvalue problem: it is necessary to find eigenvalues leading to nontrivial solutions of Eq. (2) satisfying the boundary conditions (3). The eigenvalue corresponding to the HRR problem (1) is well known $\lambda = n/(n+1)$. The further development of NFM required analysis of eigenspectra and orders of singularity at a crack tip for power-law materials [5 – 8]. In [6] the necessity of introducing higher or lower order singular terms to more correctly describe the asymptotic fields of crack tip is shown. The coordinate perturbation technique is employed to study the eigenspectra of creeping body. To attain eigensolutions a numerical scheme is worked out and the results obtained provide the information including the number of singularities, and their orders, as well as the angular distributions of stresses. The present study discusses different approaches to solve nonlinear eigenvalue problems arising in NFM for power-law materials. The main attention is paid to perturbation techniques to solve nonlinear eigenvalue problems (2), (3). Eqs. (2) and (3) form a nonlinear eigenvalue problem, where the unknown eigenvalue λ and the eigenfunction $f(\theta)$ depend on the boundary conditions and the hardening exponent. An analytical expression for the eigenvalues of the nonlinear equation can be derived by applying the perturbation method. For this purpose, the eigenvalue is split into $\lambda = \lambda_0 + \varepsilon$, where λ_0 refers to the "undisturbed" linear problem and ε is the deviation on account of the nonlinearity. The hardening exponent n and the stress function $f(\theta)$ are represented as power series

$$n = n_0 + \varepsilon n_1 + \varepsilon^2 n_2 + \varepsilon^3 n_3 + \ldots = \sum_{j=0}^{\infty} \varepsilon^j n_j, \; f(\theta) = f_0(\theta) + \varepsilon f_1(\theta) + \varepsilon^2 f_2(\theta) + \varepsilon^3 f_3(\theta) + \ldots = \sum_{j=0}^{\infty} \varepsilon^j f_j(\theta),$$

where $n_0 = 1$ and $f_0(\theta)$ are referred to the linear solution. This method allows us to find the closed form solution.

Conclusions

Recent activity is surveyed in the analysis of crack-tip stress and strain fields for stationary and growing cracks in power-law materials. Some of the main subjects to further progress are discussed. In the study the detailed review of solutions for crack problems obtained for power law constitutive equations is presented. In NFM, one often needs to solve nonlinear differential equations about eigenfunction and eigenvalue. Many nonlinear eigenvalue equations have multiple solutions. Boundary value problems of these problems are not easy to gain by means of numerical techniques such as the shooting method. The perturbation and asymptotic approximations of nonlinear problems often break down as nonlinearity becomes strong [9, 10]. Therefore, they are only valid for weakly nonlinear ordinary differential equations and partial differential equations in general. The homotopy analysis method (HAM) is an analytic approximation method for highly nonlinear problems. Unlike perturbation techniques, the HAM is independent of any small/large physical parameters at all. The HAM provides us a convenient way to guarantee the convergence of solution series so that it is valid even if nonlinearity becomes rather strong. Thus, in fracture mechanics HAM may play a significant role in solving nonlinear eigenvalue problems. The book [9] shows the great potential and validity of the for highly nonlinear eigenvalue equations with multiple solutions and singularity. The HAM will provide us one of the promising approaches for nonlinear eigenvalue problem arising in fracture mechanics. The present review shows that asymptotic solutions of fracture mechanics problem will be connected with derivation of multi-term asymptotic series expansions for the crack-tip fields using effective computer algorithms and procedures [7,8]. The further development of NFM and continuum damage mechanics will evidently be connected with experimental determination of active damage accumulation zone in the vicinity of the crack tip via interference-optic methods [11], tomographic scanning techniques [12], acoustic emission methods [13] and, undoubtedly, with using highly accurate current perturbation theory and homotopy techniques [14].

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