

# Analysis of a Singularly Perturbed Continuous Piecewise Linear System

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*Summary.* The dynamics of piecewise linear systems can often be reduced to lower dimensional invariant cones using an appropriate Poincaré map. These invariant cones can be understood as a generalization of the center manifold concept to nonsmooth systems. In this paper, we show that the singular perturbation technique applied to a slow-fast continuous piecewise linear system can deliver a good approximation of the invariant cone. The proposed approximation approach is demonstrated on an oscillator with a unilateral spring as an example of a continuous piecewise linear system in  $\mathbb{R}^3$ .

## Introduction

Recently, there has been a greater focus placed upon piecewise linear systems, due to their ability to model many complex physical phenomena. Typical applications range from mechanical systems involving dry friction [10], to neuron models [5], electronic circuits [2] and control systems [11]. Furthermore, continuous piecewise linear systems (hereafter, CPWL) are widely used to reproduce and understand various bifurcation phenomena of smooth nonlinear dynamical systems. The dynamics of CPWL systems can exhibit very interesting effects, which are impossible to observe in smooth systems [4]. A prominent example of this rich dynamic behavior was reported in [4], where the continuous matching of two stable subsystems can result in an unstable dynamics. The authors show that this behavior is possible, only if the CPWL system has an invariant cone, which is characterized by a fixed point of a corresponding Poincaré map and serves as a reduced system to investigate the stability and bifurcations of the full system. Therefore, the existence and computation of invariant cones for systems lacking smoothness are of interest. Unfortunately, the generation of invariant cones requires the numerical solution of a system of nonlinear equations and is therefore not suitable as a constructive reduction method towards a lower-dimensional dynamical model. However, the invariant cones can be understood as a generalization of center manifold theory to piecewise linear systems with an equilibrium on the switching manifold. Moreover, the long-term behavior of the full system can be described by a lower-dimensional model obtained from reducing the system to its dynamics on the invariant cone, if the latter is attractive. This perspective shows a clear similarity between invariant cones of piecewise linear systems and smooth invariant manifolds, which are used to obtain reduced order models for general differentiable systems.

For smooth nonlinear systems, projections to linear subspaces are usually used for model reduction, even though these are not invariant with respect to the original nonlinear dynamics. A reduced dynamics on an attractive invariant set, however, constitutes a mathematically more justifiable model reduction, since the trajectories of the reduced system on the invariant set are actual solutions of the full system. Relying on smoothness properties of the system, the existence and uniqueness of smooth invariant manifolds, seen as an extension of the underlying linear subspaces, have been addressed in the framework of spectral submanifolds (SSMs) [6]. The idea of model reduction in the framework of SSMs is based on a specific choice of slow variables, which determine the steady-state behavior of the system and are used as master coordinates to enslave the remaining state variables, therefore giving birth to a reduced model containing the long term characteristics of the full system. This fundamental idea emanated originally from a slow-fast decomposition using singular perturbation theory for smooth nonlinear systems. Therefore, the investigation of slow-fast CPWL systems using perturbative approximations could pave the way for the development of novel reduction methods for systems with nonsmooth nonlinearities.

The aim of this paper is to derive an approximation in closed form of the eigenvector defining the invariant half-lines of the cone for a specific homogeneous CPWL mechanical system using the theory of singular perturbations. This allows to obtain a reduced order model, for which the switching plane is modified such that the reduced dynamics is also of CPWL nature.

This paper is organized as follows. A brief overview on invariant cones along with an important result from [4] on their existence and stability are presented in next section. Then, the theory of singular perturbations is described with a focus on piecewise linear systems. In the last section, a slow-fast oscillator in  $\mathbb{R}^3$  with a unilateral spring is analyzed. An explicit expression for its invariant half-lines is derived using singular perturbation theory, and a reduced model with a modified switching condition is obtained.

## Invariant cones of continuous piecewise linear systems in $\mathbb{R}^3$

Without loss of generality, we consider a CPWL system with a single switching plane  $\Sigma = \{\mathbf{x} \in \mathbb{R}^3 : y = 0\}$  written as:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = \begin{cases} \mathbf{A}^+ \mathbf{x} & \text{for } y \geq 0 \\ \mathbf{A}^- \mathbf{x} & \text{for } y < 0 \end{cases}, \quad (1)$$

where  $\mathbf{x} = (x_1 \ x_2 \ y)^T \in \mathbb{R}^3$  and  $\mathbf{A}^\pm$  are  $3 \times 3$  real constant matrices satisfying the continuity condition  $\mathbf{A}^+ - \mathbf{A}^- = (\mathbf{A}^+ - \mathbf{A}^-) \mathbf{e}_3 \mathbf{e}_3^T$ , with  $\mathbf{e}_3$  being the third vector of the standard basis of  $\mathbb{R}^3$ . Therefore, both matrices are only different in the third column.

For the sake of brevity, we will use the compact form

$$\Upsilon(\eta) = \Upsilon^\pm = \begin{cases} \Upsilon^+ & \text{if } \eta \geq 0 \\ \Upsilon^- & \text{if } \eta < 0 \end{cases}, \quad (2)$$

where  $\Upsilon^\pm$  are either matrices or scalars. Hence, system (1) can be written as  $\dot{\mathbf{x}} = \mathbf{A}(y)\mathbf{x} = \mathbf{A}^\pm \mathbf{x}$ . The origin is always an equilibrium point of system (1), and it is the unique equilibrium if  $\mathbf{A}^\pm$  are both nonsingular. Suppose that an initial condition  $\mathbf{x}_0$  lies in one of the domains  $\mathcal{U}^\pm = \{\mathbf{x} \in \mathbb{R}^3 : y \gtrless 0\}$ , and that the corresponding trajectory remains in the same domain for any given time  $t \in (0, \infty)$  and therefore does never reach the switching plane  $\Sigma$ . In this case, the system behaves purely in a smooth fashion and the conventional theory for differentiable systems can be applied. The interesting behavior occurs, however, if the trajectory crosses  $\Sigma$  at a finite time, which can lead to various dynamical behaviors in general piecewise linear systems, such as direct crossing, sliding, grazing or jumping. In this work, we consider the class of CPWL systems, for which the dynamics can only include direct crossing behavior and the uniqueness of solutions for every initial condition is ensured. To understand the composed motion of both subsystems, we consider the following subsets of  $\Sigma$ :

$$\Sigma_{>} := \{\mathbf{x} \in \Sigma : \mathbf{e}_3^T \mathbf{A}^+ \mathbf{x} = \mathbf{e}_3^T \mathbf{A}^- \mathbf{x} > 0\}, \quad \Sigma_{<} := \{\mathbf{x} \in \Sigma : \mathbf{e}_3^T \mathbf{A}^+ \mathbf{x} = \mathbf{e}_3^T \mathbf{A}^- \mathbf{x} < 0\}$$

For  $\mathbf{x}^* \in \Sigma_{<}$  or  $\mathbf{x}^* \in \Sigma_{>}$ , the flow transitions from one domain into the other through  $\mathbf{x}^*$ . In the following, we will assume that initial values  $\mathbf{x}_0$  are chosen from the set  $\mathbf{x}_0 \in \Sigma_{<}$ . A trajectory is then given by  $\varphi(\mathbf{x}_0, t) = e^{\mathbf{A}^- t} \mathbf{x}_0$  and enters  $\mathcal{U}^+$  by means of the flow of the system  $\dot{\mathbf{x}} = \mathbf{A}^+ \mathbf{x}$ . It reaches the switching plane again for the first time at  $\mathbf{x}_1 \in \Sigma_{>}$ . Hence, there is a positive finite time  $t^-(\mathbf{x}_0) = \min\{t > 0 : \mathbf{e}_3^T e^{\mathbf{A}^- t} \mathbf{x}_0 = 0, \mathbf{e}_3^T e^{\mathbf{A}^- t} \mathbf{A}^- \mathbf{x}_0 > 0\}$ . Similarly  $t^+(\mathbf{x}_1)$  can be defined for  $\mathbf{x}_1 \in \Sigma_{>}$ . Since the flow is piecewise linear, one can see that  $t^-$  and  $t^+$  are constant on half-lines, i.e.  $t^\pm(\lambda \mathbf{x}) = t^\pm(\mathbf{x})$  with  $\lambda \in (0, \infty)$ . For initial conditions  $\mathbf{x}_0 \in \Sigma$ , the following half-maps are defined:

$$\begin{aligned} P^- : \Sigma_{<} &\rightarrow \Sigma & P^+ : \Sigma_{>} &\rightarrow \Sigma \\ \mathbf{x}_0 &\mapsto e^{\mathbf{A}^- t^-(\mathbf{x}_0)} \mathbf{x}_0 =: P^-(\mathbf{x}_0) & \mathbf{x}_1 &\mapsto e^{\mathbf{A}^+ t^+(\mathbf{x}_1)} \mathbf{x}_1 =: P^+(\mathbf{x}_1) \end{aligned}$$

Hence, the Poincaré map reads:

$$P(\mathbf{x}_0) := P^+(P^-(\mathbf{x}_0)) = e^{\mathbf{A}^+ t^+(P^-(\mathbf{x}_0))} e^{\mathbf{A}^- t^-(\mathbf{x}_0)} \mathbf{x}_0 \quad (3)$$

Since system (1) is positively homogeneous and the vector field satisfies  $\mathbf{F}(\mu \mathbf{x}) = \mu \mathbf{x}$ ,  $\forall \mathbf{x} \in \mathbb{R}^3, \mu > 0$ , the Poincaré map  $P$  transforms half-lines contained in  $\Sigma$  and passing through the origin into half-lines contained in the same plane, also passing through the origin. A more general construction is given by the following theorem stated in [7]:

**Theorem 1** *Let  $\bar{\mathbf{x}} \in \Sigma$  be an eigenvector of the nonlinear eigenvalue problem  $P(\bar{\mathbf{x}}) = \mu \bar{\mathbf{x}}$ , with some real positive eigenvalue  $\mu$ . Then there is an invariant cone for system (1). Moreover,*

- If  $\mu > 1$ , then the origin is an unstable equilibrium.
- If  $\mu = 1$ , then the cone consists of periodic orbits.
- If  $\mu < 1$ , then the stability of the origin is dependent of the stability of  $P$  w.r.t. the complimentary directions.

Therefore, system (1) has an invariant cone if there exists a half-line contained in  $\Sigma$  that is invariant for the Poincaré map  $P$ . The nonlinear eigenvalue problem  $P(\bar{\mathbf{x}}) = \mu \bar{\mathbf{x}}$  determines the invariant cone and involves the six independent variables  $\bar{\mathbf{x}} \in \mathbb{R}^3, \mu, t^-(\bar{\mathbf{x}})$  and  $t^+(P(\bar{\mathbf{x}}))$ , which can be obtained numerically as solution of the nonlinear equation system given by:

$$0 = \mathbf{G}(\mathbf{x}, t^-, t^+, \mu) = \begin{pmatrix} e^{\mathbf{A}^+ t^+} e^{\mathbf{A}^- t^-} \mathbf{x} - \mu \mathbf{x} \\ \mathbf{e}_3^T e^{\mathbf{A}^- t^-} \mathbf{x} \\ \mathbf{e}_3^T \mathbf{x} \\ \mathbf{x}^T \mathbf{x} - 1 \end{pmatrix} \quad (4)$$

This system of equations includes the definition of the Poincaré map, the first return to  $\Sigma$  at the time  $t^-$ , the location of the initial condition on  $\Sigma$  and a normalization. The zeros of  $\mathbf{G}$  can be solved numerically to obtain the 6 independent unknowns characterizing the invariant cone. Although the problem of existence and number of invariant cones in general piecewise linear systems is still open, it has been proved that there exists at most one invariant cone for some degenerate CPWL cases in [3] and [4] and at most two invariant cones for observable three dimensional CPWL cases in [1]. For the sake of completeness, we recall here an important result from [4] (Theorem 2 - Statement (a)) on the number and stability of invariant cones for a specific case, which is considered in the mechanical system studied in this work.

**Theorem 2** Suppose that system (1) satisfies the observability condition, i.e. the observability matrix

$$\mathbf{O} = \begin{pmatrix} \mathbf{e}_3^T \\ \mathbf{e}_3^T \mathbf{A}^- \\ \mathbf{e}_3^T (\mathbf{A}^-)^2 \end{pmatrix}$$

has full rank. Further, assume that the eigenvalues of matrices  $\mathbf{A}^\pm$  are  $\lambda_1^\pm = \lambda^\pm$  and  $\lambda_{2,3}^\pm = \alpha^\pm \pm i\beta^\pm$ , with  $\beta^\pm > 0$  and introduce the parameters

$$\gamma^+ = \frac{\alpha^+ - \lambda^+}{\beta^+}, \quad \text{and} \quad \gamma^- = \frac{\alpha^- - \lambda^-}{\beta^-}.$$

Then the following statement holds: If  $\gamma^+ \gamma^- > 0$ , then system (1) has only one invariant cone, which is two-zonal (i.e. lives in the two linear zones) and hyperbolic, asymptotically stable for  $\gamma^+ + \gamma^- > 0$  and unstable for  $\gamma^+ + \gamma^- < 0$ .

Note that due to the continuity condition, the observability matrix is independent of the chosen matrix,  $\mathbf{A}^+$  or  $\mathbf{A}^-$ . In the next section, a brief introduction to singular perturbation theory with an emphasis on CPWL system is given.

### Singular perturbation theory for CPWL systems

For an  $n$ -dimensional smooth system having  $s$  slow variables and a small perturbation parameter  $\varepsilon$ , which is responsible for a time-scale separation, classical geometric perturbation theory can be used to obtain a reduced-order model. The limiting case  $\varepsilon \rightarrow 0$  gives an  $f$ -dimensional critical manifold  $\mathcal{M}_c$ , where  $f = n - s$ . According to Fenichel's theorem, if  $\mathcal{M}_c$  is normally hyperbolic, then there exists an  $f$ -dimensional slow invariant manifold  $\mathcal{M}_s$ , on which the dynamics is a perturbation of order  $\mathcal{O}(\varepsilon)$  of the dynamics on  $\mathcal{M}_c$ . This theorem can be applied to slow-fast CPWL systems only on the subsets of the state space that do not include the switching manifold. This yields two linear locally invariant slow half-manifolds  $\mathcal{M}_s^\pm$ , each aligned with the slow eigenspaces of  $\mathbf{A}^+$  or  $\mathbf{A}^-$ . Furthermore, a forward invariant neighborhood enveloping the linear critical manifold, which is continuous at the switching manifold, has been shown to exist under suitable conditions [9]. For this, the critical manifold  $\mathcal{M}_c$ , which is not normally hyperbolic on  $\Sigma$ , has to be attracting. Consider a slow-fast ODE system of the form:

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{cases} \mathbf{f}^+(\mathbf{x}, \mathbf{y}; \varepsilon) & \text{for } h(\mathbf{x}, \mathbf{y}) \geq 0 \\ \mathbf{f}^-(\mathbf{x}, \mathbf{y}; \varepsilon) & \text{for } h(\mathbf{x}, \mathbf{y}) < 0 \end{cases} \\ \varepsilon \dot{\mathbf{y}} &= \begin{cases} \mathbf{g}^+(\mathbf{x}, \mathbf{y}; \varepsilon) & \text{for } h(\mathbf{x}, \mathbf{y}) \geq 0 \\ \mathbf{g}^-(\mathbf{x}, \mathbf{y}; \varepsilon) & \text{for } h(\mathbf{x}, \mathbf{y}) < 0, \end{cases} \end{aligned} \quad (5)$$

where  $\mathbf{x} = (x_1 \ \cdots \ x_s)^T \in \mathbb{R}^s$  are the slow variables,  $\mathbf{y} = (y_1 \ \cdots \ y_f)^T \in \mathbb{R}^f$  are the fast variables,  $0 \leq \varepsilon \ll 1$  is the small parameter and  $\dot{(\cdot)} := \frac{d(\cdot)}{dt}$  denotes the derivative with respect to the "slow" time scale  $t$ . The switching manifold is therefore given by the scalar function  $h(\mathbf{x}, \mathbf{y}) = 0$ . We assume that the functions  $\mathbf{f}^\pm, \mathbf{g}^\pm$  and  $h$  are linear with respect to  $\mathbf{x}$  and  $\mathbf{y}$  and that the system is continuous at the switching manifold. Furthermore, suppose that  $h(\mathbf{0}, \mathbf{0}) = 0$ . At the origin, we also assume that the switching manifold is not tangent to all fast directions, which means that  $\nabla h$  has at least one non-zero component. Without loss of generality, one can assume  $\frac{\partial h}{\partial y_1} \neq 0$ . This assumption leads to a more general configuration, where both the slow and fast dynamics contain a switch. Otherwise, a degenerate system is obtained, where only the slow dynamics has a switch and the fast dynamics is  $\mathbf{g}^\pm = \mathbf{g}$ . In order to simplify the switching condition, an invertible transformation  $(\mathbf{x}, \mathbf{y}) \rightarrow (\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$  is introduced, with a new set of coordinates  $\tilde{\mathbf{x}} = \mathbf{x}$  and  $\tilde{\mathbf{y}} = (h(\mathbf{x}, \mathbf{y}) \ y_2 \ y_3 \ \cdots \ y_f)^T$ . Since  $\tilde{y}_1$  is a new fast variable, the slow-fast system has the same form as (5) except that the switching manifold is determined by  $\tilde{y}_1 = 0$ . Taking the new coordinates and setting the small parameter  $\varepsilon = 0$  in (5) yields the critical system

$$\dot{\mathbf{x}} = \mathbf{f}^\pm(\mathbf{x}, \mathbf{y}; 0) \quad (6)$$

$$0 = \mathbf{g}^\pm(\mathbf{x}, \mathbf{y}; 0) \quad (7)$$

where the  $\tilde{(\cdot)}$  is dropped for simplicity, and the  $\pm$  switch is governed by equation (2) with  $y_1 = \eta$ . The critical manifold is obtained as  $\mathcal{M}_c = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{s+f} : \mathbf{y} = \mathbf{h}_c^\pm(\mathbf{x})\}$ , where  $\mathbf{y} = \mathbf{h}_c^\pm(\mathbf{x})$  are the solutions of the two algebraic constraints (7) and describe the behavior of the fast variables as a function of the slow variables. Note that  $\mathbf{h}_c^\pm$  are both linear functions of  $\mathbf{x}$  and that the matching from both linear subsystems is continuous. The dynamics on  $\mathcal{M}_c$  is governed by

$$\dot{\mathbf{x}} = \mathbf{f}^\pm(\mathbf{x}, \mathbf{h}_c^\pm(\mathbf{x}); 0). \quad (8)$$

The Jacobians  $\left. \frac{\partial \mathbf{g}^\pm}{\partial \mathbf{y}}(\mathbf{x}) \right|_{\mathbf{y}=\mathbf{h}_c^\pm, \varepsilon=0}$  along both critical manifolds are assumed to fulfill the stability condition, so that a relevant reduction to the slow dynamics can be obtained. For CPWL systems, the existence of a forward invariant neighborhood around the critical manifold has been shown in [9], if  $\mathcal{M}_c$  is globally exponentially stable. Therefore, and for the

sake of brevity, we assume global exponential stability of the critical manifold, which is naturally given for the specific example considered in this work. We refer to [9] for more details on the stability properties of  $\mathcal{M}_c$  and the proof of the existence of a forward invariant neighborhood. For systems of the form (1), the singular perturbation technique is performed on each linear subsystem to obtain the linear locally invariant slow half-manifolds. This is described in the following, where the  $\pm$  switching is dropped for simplicity and only the linear region defined by  $y_1 \geq 0$  is considered. Obviously, the approach is analogous for the other linear region. The matching of both linear invariant half-manifolds and the corresponding switching condition are discussed explicitly for the example in the next section.

The  $s$ -dimensional, locally invariant slow manifold is defined as  $\mathcal{M}_s^+ = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{s+f} : \mathbf{y} = \mathbf{h}_s^+(\mathbf{x})\}$ . Since the state space is decomposed into two linear parts, the invariance property of  $\mathcal{M}_s^+$  must be understood in a local way. Inserting  $\mathbf{y} = \mathbf{h}_s^+$  and  $\dot{\mathbf{y}} = \frac{\partial \mathbf{h}_s^+}{\partial \mathbf{x}} \Big|_{\mathbf{x}, \varepsilon} \dot{\mathbf{x}}$  into the fast dynamics  $\varepsilon \dot{\mathbf{y}} = \mathbf{g}^+(\mathbf{x}, \mathbf{y}, \varepsilon)$  yields:

$$\varepsilon \frac{\partial \mathbf{h}_s^+}{\partial \mathbf{x}} \mathbf{f}^+(\mathbf{x}, \mathbf{h}_s^+; \varepsilon) = \mathbf{g}^+(\mathbf{x}, \mathbf{h}_s^+; \varepsilon). \quad (9)$$

In each linear region, the asymptotic expansion given by

$$\mathbf{h}_s^+(\mathbf{x}) = \mathbf{h}_0^+(\mathbf{x}) + \varepsilon \mathbf{h}_1^+(\mathbf{x}) + \mathcal{O}(\varepsilon^2) \quad (10)$$

is used in the invariance equation (9). By equating the coefficients of powers of  $\varepsilon$  one can see that  $\mathbf{h}_c^+(\mathbf{x}) = \mathbf{h}_0^+(\mathbf{x})$ , which means that the critical manifold is the zero-order approximation of the slow manifold. Moreover, the first order term  $\mathbf{h}_1^+(\mathbf{x})$  is obtained as:

$$\mathbf{h}_1^+ = \frac{\partial \mathbf{g}^+}{\partial \mathbf{y}} \Big|_{\mathbf{y}=\mathbf{h}_0^+; \varepsilon=0}^{-1} \left[ \frac{\partial \mathbf{h}_0^+}{\partial \mathbf{x}} \mathbf{f}^+(\mathbf{x}, \mathbf{h}_0^+; 0) - \frac{\partial \mathbf{g}^+}{\partial \varepsilon} \Big|_{\mathbf{x}, \mathbf{h}_0^+; 0} \right]. \quad (11)$$

### Three-dimensional oscillator with a unilateral spring

The approximation of the invariant cone by means of singular perturbation theory is demonstrated on the CPWL system shown in Figure 1. The system consists of a mass  $m$  and a massless rod, each coupled to the environment by a spring-damper element. In addition, the mass is connected to the rod by one linear and one unilateral spring, with stiffnesses  $k_3$  and  $k_{N3}$ , respectively. The unilateral spring is active only when the relative displacement  $q_1 - q_2$  is positive, where  $q_1$  and  $q_2$  are the displacements of the mass and the massless rod, respectively. Let  $\mathbf{x} = (x_1 \ x_2 \ y)^T \in \mathbb{R}^3$  be the state vector, where the components are defined as follows:

$$x_1 = q_1, \quad x_2 = \dot{q}_1, \quad y = q_1 - q_2. \quad (12)$$

In this set of coordinates, the equations of motion have the form of system (1), where  $\mathbf{A}^\pm$  are constant matrices given by

$$\mathbf{A}^\pm = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{k_1}{m} & -\frac{c_1}{m} & -\frac{k_3^\pm}{m} \\ \frac{k_2}{c_2} & 1 & -\frac{k_2+k_3^\pm}{c_2} \end{pmatrix} \quad \text{with} \quad \begin{cases} k_3^+ = k_3 + k_{N3} & \text{for } y \geq 0 \\ k_3^- = k_3 & \text{for } y < 0 \end{cases}, \quad (13)$$

with all damping and stiffness coefficients assumed to be non-negative. The switching manifold is defined as  $\Sigma = \{\mathbf{x} \in \mathbb{R}^3 | y = 0\}$  and the state space consists of two half-spaces  $\mathcal{U}^\pm = \{\mathbf{x} \in \mathbb{R}^3 | y \gtrless 0\}$ . By statement (a) of Theorem 2 and after performing simple calculations for fixed sets of parameters which fulfill the assumptions on the matrices  $\mathbf{A}^\pm$ , we deduce that this system has only one invariant cone which is hyperbolic and asymptotically stable. As the existence of a stable invariant cone is now established, the next step is to apply singular perturbation theory to approximate the cone. In order to bring the system to a singularly perturbed form, the damping constant  $c_2$  is assumed to be a small parameter ( $c_2 = \varepsilon$ ). Next, we split the state vector  $\mathbf{x}$  into slow variables  $\mathbf{x}_s = (x_1 \ x_2)^T \in \mathbb{R}^2$  and a scalar fast variable  $y$ .

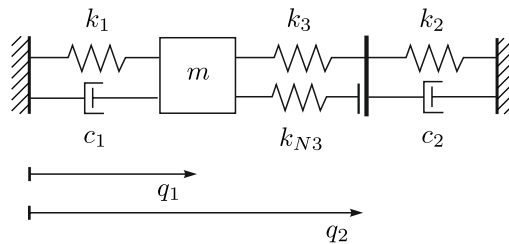


Figure 1: Mechanical system of a slow-fast oscillator in  $\mathbb{R}^3$ .

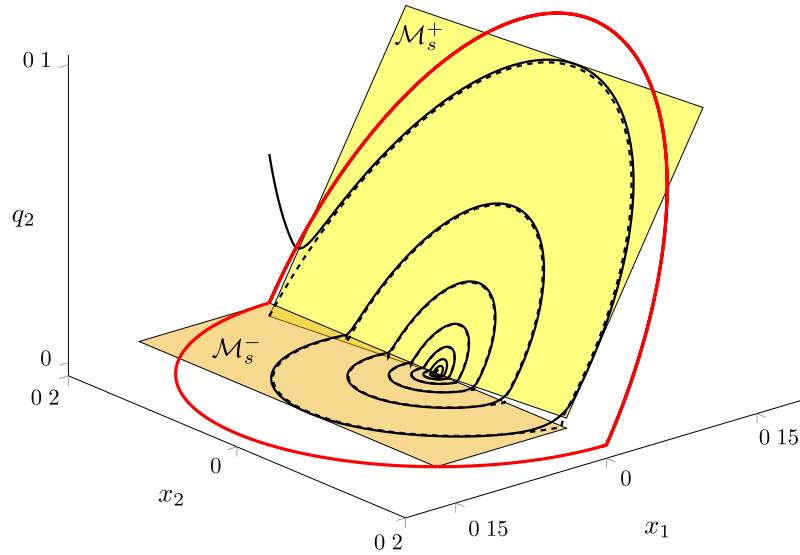


Figure 2: Illustration of the attracting invariant cone approximated by two half-planes  $\mathcal{M}_s^\pm$  with a switch at  $x_1 = 0$  for the parameter set  $\varepsilon = c_2 = 0.1, c_1 = 0.4, m = 1, k_1 = k_2 = 1, k_3 = 0, k_{N3} = 2$ . A general trajectory of the full system (solid line) is attracted to the half-planes and synchronizes with a trajectory of reduced dynamics (dashed line). The critical manifold gives a conservative periodic orbit (red line) and is therefore not suitable as an approximation of the full system.

The equations of motion in the slow-fast form read as:

$$\dot{\mathbf{x}}_s = \mathbf{f}^\pm(\mathbf{x}_s, y) = \begin{pmatrix} x_2 \\ -\frac{k_1}{m}x_1 - \frac{c_1}{m}x_2 - \frac{k_3^\pm}{m}y \end{pmatrix} \quad (14)$$

$$\varepsilon \dot{y} = g^\pm(\mathbf{x}_s, y, \varepsilon) = k_2x_1 + \varepsilon x_2 - (k_2 + k_3^\pm)y. \quad (15)$$

The critical manifolds in the half-spaces  $\mathcal{U}^\pm$  are obtained as isolated solutions  $h_0^\pm(\mathbf{x}_s)$  of the equations  $g^\pm(\mathbf{x}_s, h_0^\pm(\mathbf{x}_s), 0) = 0$  and read as:

$$h_0^\pm(\mathbf{x}_s) = \frac{k_2}{k_2 + k_3^\pm}x_1 \quad (16)$$

Herein,  $h_0^+(\mathbf{x}_s)$  is only applicable for  $h_0^+(\mathbf{x}_s) \geq 0$ , which in view of  $k_2, k_3^+ \geq 0$  comes down to  $x_1 \geq 0$ . Similarly  $h_0^-(\mathbf{x}_s)$  is only applicable for  $x_1 < 0$ . Hence, we may define the critical manifold by

$$h_0(\mathbf{x}_s) = \begin{cases} h_0^+(\mathbf{x}_s) & \text{for } x_1 \geq 0 \\ h_0^-(\mathbf{x}_s) & \text{for } x_1 < 0 \end{cases} \quad (17)$$

The Jacobians  $\left. \frac{\partial g^\pm}{\partial y}(\mathbf{x}_s) \right|_{y=h_0^\pm, \varepsilon=0}$  along the critical manifold are strictly negative and fulfill the stability condition, and the dynamics on this manifold is given by  $\dot{\mathbf{x}}_s = \mathbf{f}^\pm(\mathbf{x}_s, y = h_0(\mathbf{x}_s); 0)$ . For the special choice  $c_1 = 0$ , this dynamics is purely conservative and yields a periodic orbit, which does not reflect the dissipative nature of the original system ( $\varepsilon \neq 0$ ) and therefore cannot be used to approximate the long time behavior of the full system, as shown in Figure 2. Thus, terms of  $\mathcal{O}(\varepsilon)$  must be included to obtain a dissipative reduced-order model. Using the equations (10) and (11), the locally invariant slow half-manifolds of the two linear subsystems read as:

$$y_{\text{slow}} = h_s^\pm(\mathbf{x}_s) = \frac{k_2}{k_2 + k_3^\pm}x_1 + \frac{\varepsilon k_3^\pm}{(k_2 + k_3^\pm)^2}x_2 + \mathcal{O}(\varepsilon^2), \quad \text{with } k_3^\pm = \begin{cases} k_3 + k_{N3} & \text{for } y_{\text{slow}} \geq 0 \\ k_3 & \text{for } y_{\text{slow}} < 0 \end{cases}. \quad (18)$$

The dynamics of the 2-dimensional system, reduced to the linear locally invariant slow manifolds in both regions, is governed by  $\dot{\mathbf{x}}_s = \mathbf{f}^\pm(\mathbf{x}_s, h^\pm(\mathbf{x}_s))$ . The switching condition in this case is not trivial anymore, since the stiffness  $k_3^\pm$  depends on  $y_{\text{slow}}$  itself. This problem is illustrated in Figure 3. The colored areas and their corresponding limits are obtained from (18) by solving the inequalities  $y_{\text{slow}} \geq 0$  (yellow region for  $h_s^+$ ) and  $y_{\text{slow}} < 0$  (orange region for  $h_s^-$ ), which matches the physical switching condition. In the white area, none of the inequalities are satisfied, whereas the dark area shows the region where both inequalities are fulfilled. To circumvent this problem of switching between  $h_s^\pm$ , one could take  $x_1 = 0$  as a switching plane, since the  $x_2$  terms in (18) are of order  $\varepsilon$ . At this modified switching plane, the linear slow manifolds, obtained as two half-planes  $\mathcal{M}_s^\pm = \{\mathbf{x} \in \mathbb{R}^3 | y = h_s^\pm(\mathbf{x}_s)\}$ , are askew and meet only at the equilibrium point. This leads to a reduced system containing a jump at  $x_1 = 0$ . However, the trajectories of the reduced system are still a good approximation and follow the trajectories of the full system, which converge asymptotically towards  $\mathcal{M}_s^\pm$  as shown in Figure 2. In order to obtain a continuous reduced system,  $\mathcal{M}_s^\pm$  can be continued up to their intersection line. This leads to the avoidance of jumps in the reduced dynamics at  $\Sigma$ .

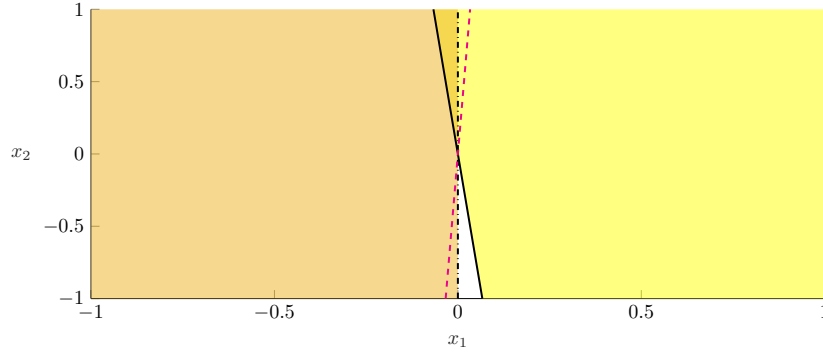


Figure 3: Illustration of the admissibility regions for  $h_s^+$  (yellow area with the solid black line as limit) and  $h_s^-$  (orange area with the dash-dotted black line as limit). The white area is the region for which both  $h_s^\pm$  are not admissible. The magenta dashed line shows the intersection of the two linear slow manifolds. Example for the parameter set  $\varepsilon = c_2 = 0.1, c_1 = 0, m = 1, k_1 = k_2 = 1, k_3 = 0, k_{N3} = 2$ .

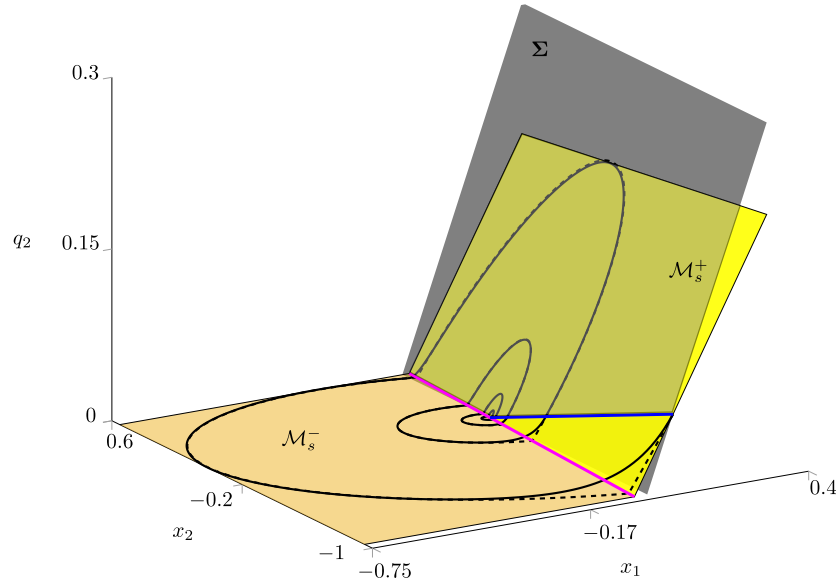


Figure 4: Illustration of the invariant cone for  $\varepsilon = c_2 = 0.1, c_1 = 0.4, m = 1, k_1 = k_2 = 1, k_3 = 0, k_{N3} = 2$ . The magenta line shows the intersection line of the two locally invariant half-planes. The blue line is the intersection line of  $\mathcal{M}_s^+$  with the switching manifold  $\Sigma$ , which lies exactly on the direction of the eigenvector from the numerical solution of equation (4).

### Continued reduced slow dynamics

The main disadvantage of taking  $x_1 = 0$  as a switching plane in (18) is that the reconstruction of the fast variable  $y$  contains a jump leading to a Filippov system [8] for the reduced dynamics. For the purpose of obtaining a continuous matching of the two locally invariant half-planes, the switching plane can be modified and fixed at the intersection line defined by  $\mathcal{M}_s^+ \cap \mathcal{M}_s^-$ . Setting  $h_s^+ = h_s^-$  yields:

$$\left(-\frac{k_2}{k_2 + k_3^+} + \frac{k_2}{k_2 + k_3^-}\right)x_1 + \left(\frac{k_3^-}{(k_2 + k_3^-)^2} - \frac{k_3^+}{(k_2 + k_3^+)^2}\right)x_2 = 0. \quad (19)$$

This condition  $h_s^+ = h_s^-$  is then used in (18) as switching condition instead of  $y_{\text{slow}}$ . Even though this line does not lie on the physical switching plane  $\Sigma$ , defined by  $y = 0$ , the reduced dynamics gives a better approximation of the dynamics of the full system. This is illustrated in Figure 4, where the reduced dynamics on the slow manifolds (dashed line) contains only a kink at the switching instead of a jump. The trajectory of the reduced system follows the solution of the full system more closely than trajectories containing a jump at  $x_1 = 0$ . Figure 5 compares the time histories of the displacements  $q_1$  and  $q_2$  over some periods of decaying oscillations from the full system (black), the reduced system with a jump (blue) and the reduced model from the continued invariant half-planes (red). The continued reduced dynamics shows a closer agreement with the full system than the reduced model containing jumps.

### Approximation of the invariant half-lines

Since the global stability of the continuous critical manifold  $\mathcal{M}_c$  is ensured and the two slow manifolds  $\mathcal{M}_s^\pm$  are locally invariant and attracting, a typical trajectory in  $\mathcal{U}^+$  would cross  $\Sigma$  from right to left at a point that can be assumed in the  $\mathcal{O}(\varepsilon)$  neighborhood of  $\mathcal{M}_s^+$ . For sufficiently small  $\varepsilon$ , the trajectory would then approach  $\mathcal{M}_s^-$  and follow it closely

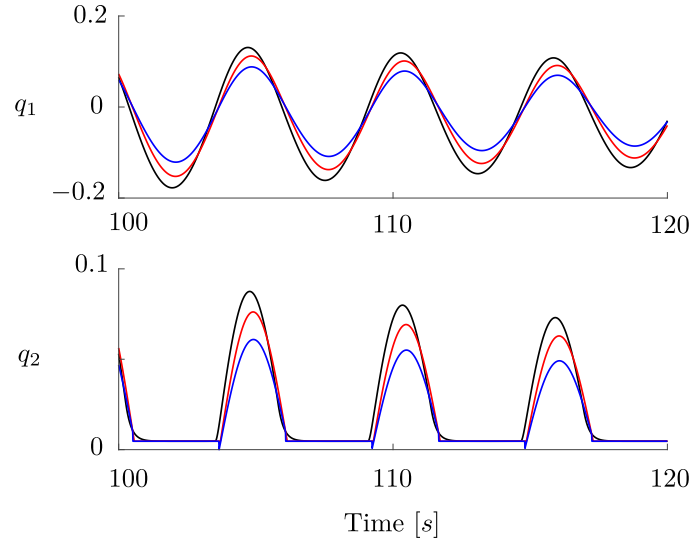


Figure 5: Time histories for  $\varepsilon = c_2 = 0.2$ ,  $c_1 = 0$ ,  $m = 1$ ,  $k_1 = k_2 = 1$ ,  $k_3 = 0$ ,  $k_{N3} = 2$ . The black, blue and red lines show the displacements from the full system, the reduced system with a jump and the continued reduced system, respectively.

until the next crossing with  $\Sigma$ . Hence, we can assume that every transition from right to left through  $\Sigma$  is in a  $\mathcal{O}(\varepsilon)$  neighborhood of  $\mathcal{M}_s^+$ . Consequently, the intersection line of  $\mathcal{M}_s^+$  and  $\Sigma$  presents a good approximation of an invariant half-line of the invariant cone and can be obtained by setting  $y_{\text{slow}}^+ = 0$  in (18) as :

$$\mathbf{r}_{\text{inv}} = \{x_1 = -\varepsilon \frac{k_3^+}{k_2(k_2 + k_3^+)} x_2, y = 0\}. \quad (20)$$

Using the definitions in (12), this half-line is visualized as the blue line in Figure 4 in the original coordinates of the system  $(q_1, \dot{q}_1, q_2)^T$ . For  $\varepsilon$  sufficiently small, this simple closed form approximation matches the invariant half-lines obtained from the numerical solution of (4). Obviously, this argumentation is analogous if we assume trajectories starting near the intersection of  $\mathcal{M}_s^-$  and  $\Sigma$  and leads to an approximation of the other invariant half-line, which corresponds to a Poincaré map defined as  $P = P^-(P^+)$ .

## Conclusion

This study set out to explore the connection between singular perturbation theory applied to a slow-fast CPWL system and its invariant cones. The results show that the invariant half-lines of the corresponding invariant cone can directly be approximated in a closed form from a geometric perspective by computing the intersection of the locally invariant slow manifolds and the switching plane. Moreover, this work has also highlighted that trajectories of the reduced dynamics with a modified switching condition, which was obtained from the intersection line of the two slow manifolds, can closely approximate trajectories of the full system. Further research might explore the applicability of these findings for higher dimensional CPWL systems.

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