Shocks and solitary waves in series connected discrete Josephson transmission lines

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Summary. We analytically study the running waves propagation in the discrete Josephson transmission lines (JTL), constructed from Josephson junctions (JJ) and capacitors. Due to the competition between the intrinsic dispersion and the nonlinearity, in the dissipationless JTL there exist running waves in the form of supersonic kinks and solitons. The velocities and the profiles of the kinks and the solitons are found. We also study the effect of dissipation in the system and find that in the presence of the resistors, shunting the JJ and/or in series with the ground capacitors, the only possible stationary running waves are the shock waves, whose velocities and the profiles are also found.

Introduction

The concept that in a nonlinear wave propagation system the various parts of the wave travel with different velocities, and that wave fronts (or tails) can sharpen into shock waves, is deeply imbedded in the classical theory of fluid dynamics [1]. The methods developed in that field can be profitably used to study signal propagation in nonlinear transmission lines [2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. In the early studies of shock waves in transmission lines, the origin of the nonlinearity was due to nonlinear capacitance in the circuit [12, 13, 14].

Interesting and potentially important examples of nonlinear transmission lines are circuits containing Josephson junctions (JJ) [15] - Josephson transmission lines (JTL) [16, 17, 18, 19]. The unique nonlinear properties of JTL allow to construct soliton propagators, microwave oscillators, mixers, detectors, parametric amplifiers, and analog amplifiers [17, 19, 18].

Transmission lines formed by JJ connected in series were studied beginning from 1990s, though much less than transmission lines formed by JJ connected in parallel [20]. However, the former began to attract quite a lot of attention recently [21, 22, 23, 24, 25, 26, 27, 28], especially in connection with possible JTL traveling wave parametric amplification [29, 30, 31].

The interest in studies of discrete nonlinear electrical transmission lines, in particular of lossy nonlinear transmission lines, has started some time ago [32, 33, 34], but it became even more pronounced recently [35, 36, 37]. These studies should be seen in the general context of waves in strongly nonlinear discrete systems [38, 39, 40, 41, 42, 43, 44].

In our previous publication [45] we considered shock waves in the continuous JTL with resistors, studying the influence of those on the shock profile. Now we want to analyse wave propagation in the discrete JTL, both lossless and lossy

The rest of the paper is constructed as follows. In Section we formulate quasi-continuum approximation for the discrete lossless JTL. In Section we show that the problem of a running wave is reduced to an effective mechanical problem, describing motion of a fictitious particle. In Section the velocity and the profile of the kink, and in Section - of the soliton are found from the solution of the effective mechanical problem. In Section we rigorously justify the quasi-continuum approximation for the kinks and solitons in certain limiting cases. In Section we discuss the effect of dissipation on the waves propagation in the discrete JTL. In Section we briefly mention possible applications of the results obtained in the paper and opportunities for their generalization. We conclude in Section . In the Appendix we propose the integral approximation to the discrete equations.

The quasi-continuum approximation

Consider the model of JTL constructed from identical JJ and capacitors, which is shown on Fig. 1. We take as dynamical variables the phase differences (which we for brevity will call just phases) φ_n across the JJ and the charges q_n which have passed through the JJ. The circuit equations are

$$\frac{\hbar}{2e}\frac{d\varphi_n}{dt} = \frac{1}{C}\left(q_{n+1} - 2q_n + q_{n-1}\right) , \tag{1a}$$

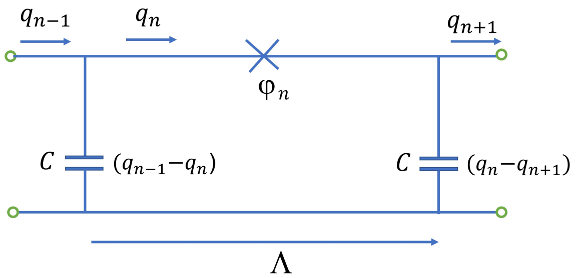
$$\frac{dq_n}{dt} = I_c \sin\varphi_n , \tag{1b}$$

where C is the capacitor, and I_c is the critical current of the JJ.

Everywhere in this paper we'll treat $q_n(t)$ ($\varphi_n(t)$) as a function of two continuous variables (z,t), where $z = n\Lambda$, and will make the simplest assumption,

$$q_{n+1} - 2q_n + q_{n-1} = \Lambda^2\frac{\partial^2 q}{\partial z^2} + \frac{\Lambda^4}{12}\frac{\partial^4 q}{\partial z^4} . \tag{2}$$

(To keep in (2) only the first term would be an even simpler assumption, but the effects we'll be talking about are absent in this approximation.) We will call (2) the quasi-continuum approximation and will se later that in certain limiting cases it can be rigorously justified.



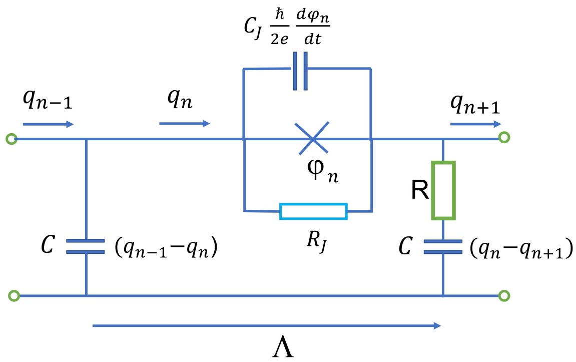


Figure 1: Discrete JTL (left) and discrete JTL with the capacitor and the resistor shunting the JJ and another resistor in series with the ground capacitor (right).

Newtonian equation

The running wave solutions are of the form

$$\varphi(z,t) = \varphi(x)\,, \qquad q(z,t) = q(x)\,, \tag{3}$$

where $x = Ut - z$, and U is the running wave velocity. For such solutions, and after the truncation, Eq. (1) becomes the ordinary differential equation

$$\overline{U}^2\frac{d\varphi}{dx} = \frac{d\sin\varphi}{dx} + \frac{\Lambda^2}{12}\frac{d^3\sin\varphi}{dx^3}\,; \tag{4}$$

in this paper, for any velocity V, $\overline{V} \equiv V\sqrt{L_JC}/\Lambda$, and $L_J = \hbar/(2eI_c)$. Integrating with respect to x we obtain

$$\frac{\Lambda^2}{12}\frac{d^2\sin\varphi}{dx^2} = -\sin\varphi + \overline{U}^2\varphi + F\,, \tag{5}$$

where F is the constant of integration. Multiplying Eq. (5) by $d\sin\varphi/dx$ and integrating once again we obtain

$$\frac{\Lambda^2}{24}\left(\frac{d\sin\varphi}{dx}\right)^2 + \Pi(\sin\varphi) = E\,, \tag{6}$$

where

$$\Pi(\sin\varphi) = \frac{1}{2}\sin^2\varphi - \overline{U}^2(\varphi\sin\varphi + \cos\varphi) - F\sin\varphi\,, \tag{7}$$

and E is another constant of integration. Equation (6) can be integrated in quadratures in the general case.
We can think about x as time and about $\sin\varphi$ as coordinate of the fictitious particle, thus considering (5) as the Newtonian equation. We are interested in the propagation of the waves characterised by the boundary conditions

$$\lim_{x\to-\infty}\varphi = \varphi_1\,, \qquad \lim_{x\to+\infty}\varphi = \varphi_2\,, \tag{8}$$

Thus the problem of finding the profile of the wave is reduced to studying the motion of the particle which starts from an equilibrium position, and ends in an equilibrium position.
Using the expertise we acquired in mechanics classes, we come to the conclusion that the initial position corresponds to maxima of the "potential energy" $\Pi(\sin\varphi)$, and so does the final position. Either these are two different maxima, or the same maximum. In the latter case the particle returns to the initial position after reflection from a potential wall. (See Figs. 2 (above) and 3.) In the first case the solution describes the kink, in the second - the soliton.
One should compare the running wave velocity with the velocity $u(\varphi_1)$ of propagation along the JTL of small amplitude smooth disturbances of φ on a homogeneous background φ_1 [45]

$$\overline{u}^2(\varphi_1) = \cos\varphi_1 \tag{9}$$

(in this paper we consider only the solutions which lie completely in the sector $(-\pi/2,\pi/2)$.) From the fact that there is a maximum of the "potential energy" at the point φ_1, follows that

$$\left.\frac{d^2\Pi(\varphi)}{d\varphi^2}\right|_{\varphi=\varphi_1} < 0\,. \tag{10}$$

Calculating the derivatives we obtain

$$\overline{U}^2 > \cos\varphi_1\,, \tag{11}$$

that is the running wave is supersonic.

The kinks

In the case of the kink, going in Eq. (5) to the limits $x \to +\infty$ and $x \to -\infty$ we obtain

$$\overline{U}^2 \varphi_1 = \sin\varphi_1 - F\,, \tag{12a}$$
$$\overline{U}^2 \varphi_2 = \sin\varphi_2 - F\,. \tag{12b}$$

Solving (12) relative to $\overline{U}^2$ and F we obtain we obtain

$$\overline{U}^2 = \frac{\sin\varphi_1 - \sin\varphi_2}{\varphi_1 - \varphi_2} \equiv \overline{U}^2_{\rm sh}(\varphi_1, \varphi_2)\,, \tag{13a}$$
$$F = \frac{\varphi_1 \sin\varphi_2 - \varphi_2 \sin\varphi_1}{\varphi_1 - \varphi_2}\,; \tag{13b}$$

the reason, why we have chosen subscript sh for the velocity in (13a), will become clear in Section .
The result for the kink velocity (13a) is more robust than it looks. In fact, summing up (1a) from far to the left of the kink up to far to the right of the kink we obtain

$$\frac{\hbar}{2e}\frac{d}{dt}\sum_n \varphi_n = \frac{1}{C}\left[(q_{n+1} - q_n)_1 - (q_{n+1} - q_n)_2\right]\,. \tag{14}$$

From the running wave ansatz follows

$$\frac{d}{dt}\sum_n \varphi_n = \frac{U}{\Lambda}\left(\varphi_1 - \varphi_2\right)\,. \tag{15}$$

To deal with the r.h.s. of (14) we need to approximate the finite difference only far away from the kink, where everything changes slowly, and the continuum approximation

$$q_{n+1} - q_n = \Lambda\frac{\partial q}{\partial z} \tag{16}$$

is enough. From (16) and the running wave ansatz follows

$$(q_{n+1} - q_n)_i = \frac{\Lambda}{U}\left(\frac{dq_n}{dt}\right)_i = \frac{\Lambda}{U}\sin\varphi_i\,. \tag{17}$$

Substituting into (14) we recover (13a).
Returning to (13) and taking into account additionally the equality

$$E = \Pi(\sin\varphi_1) = \Pi(\sin\varphi_2)\,, \tag{18}$$

we obtain

$$\varphi_2 \quad = -\varphi_1\,. \tag{19}$$

Thus the kinks which can propagate in JTL are very special. We also obtain

$$F = 0\,, \tag{20a}$$
$$\overline{U}^2 = \overline{U}^2_{\rm sh}(\varphi_1, -\varphi_1) = \frac{\sin\varphi_1}{\varphi_1} \equiv \overline{U}^2_k(\varphi_1)\,. \tag{20b}$$
$$\Pi(\sin\varphi) - E = \frac{1}{2}(\sin\varphi - \sin\varphi_1)^2 - \frac{\sin\varphi_1}{\varphi_1}\left[\cos\varphi - \cos\varphi_1 - (\varphi_1 - \varphi)\sin\varphi\right]. \tag{20c}$$

Equation (20c) and the results of integration of Eq. (6) for this "potential energy" are graphically presented on Fig. 2 (above).
Consider specifically the limiting case $|\varphi_1| \ll 1$. Expanding the "potential energy" with respect to φ and φ_1 and keeping only the lowest order terms we obtain the approximation to Eq. (6) in the form

$$\Lambda^2\left(\frac{d\varphi}{dx}\right)^2 = \left(\varphi_1^2 - \varphi^2\right)^2\,. \tag{21}$$

The solution of Eq. (21) is

$$\varphi(x) = -\varphi_1 \tanh\frac{\varphi_1 x}{\Lambda}\,. \tag{22}$$

Equations (22) coincides with that obtained by Katayama et al. [36]. So does Eq. (20b), being expanded in series with respect to φ_1 and truncated after the first two terms:

$$\overline{U}^2_k(\varphi_1) = 1 - \frac{\varphi_1^2}{6}. \tag{23}$$

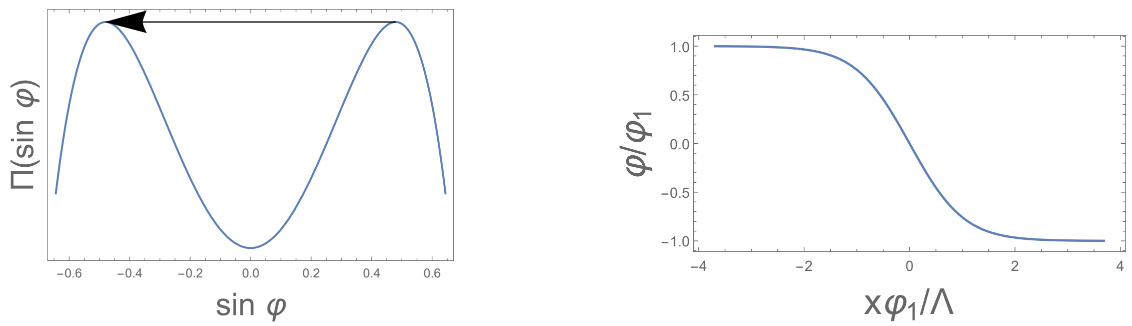


Figure 2: The "potential energy" (20c) (left) and the kink profile calculated with this energy according to Eq. (6) (right). We have chosen $\varphi_1 = .5$.

The solitons

For the soliton $\varphi_2 = \varphi_1$, and two equations of (12) become one equation. As an additional parameter we take the amplitude of the soliton (maximally different from φ_1 value of φ), which we will designate as φ_0. Adding to (12) the equation

$$E = \Pi(\sin\varphi_0) = \Pi(\sin\varphi_1) \tag{24}$$

and solving the obtained system we obtain

$$\overline{U}^2_{sol}(\varphi_1,\varphi_0) = \frac{(\sin\varphi_1-\sin\varphi_0)^2}{2[\cos\varphi_0-\cos\varphi_1-(\varphi_1-\varphi_0)\sin\varphi_0]}\,, \tag{25a}$$

$$\Pi(\sin\varphi)-E = \frac{1}{2}\left(\sin\varphi_1-\sin\varphi\right)^2 - \overline{U}^2_{sol}(\varphi_1,\varphi_0)\left[\cos\varphi-\cos\varphi_1-(\varphi_1-\varphi)\sin\varphi\right]\,. \tag{25b}$$

Equation (25b) is graphically presented on Fig. 3 (left).

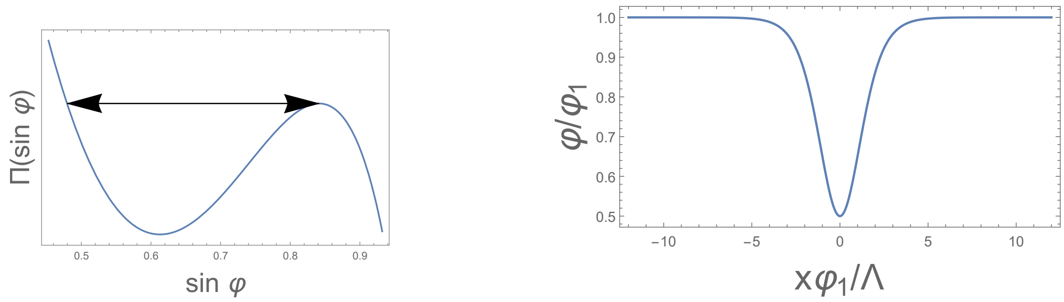


Figure 3: The "potential energy" (25b) (left) and the soliton profile according to Eq. (27) (right)

Considering the limiting case $|\varphi_1|, |\varphi_0| \ll 1$, expanding Eq. (25b) with respect to all the phases and keeping only the lowest order terms we obtain Eq. (6) in the form

$$\Lambda^2\left(\frac{d\varphi}{dx}\right)^2 = (\varphi-\varphi_1)^2(\varphi-\varphi_0)(\varphi+2\varphi_1+\varphi_0)\,. \tag{26}$$

Equation (26) can be integrated in elementary functions

$$\varphi = \varphi_1 - \frac{(4\overline{\varphi}+\Delta\varphi)\Delta\varphi}{4\overline{\varphi}\cosh^2\Phi+\Delta\varphi}\,, \tag{27}$$

where $\Delta\varphi \equiv \varphi_1-\varphi_0$, $\overline{\varphi} \equiv (\varphi_1+\varphi_0)/2$, $\Phi \equiv \sqrt{(3\varphi_1+\varphi_0)\Delta\varphi}\;x/(2\Lambda)$. Equation (27) is graphically presented on Fig. 3 (right).

In an another limiting case of weak soliton ($\Delta\varphi\cot\varphi_1 \ll 1$), Eq. (6) takes the form

$$\Lambda^2\left(\frac{d\varphi}{dx}\right)^2 = 4\tan\varphi_1\cdot(\varphi-\varphi_1)^2(\varphi-\varphi_0)\,. \tag{28}$$

The solution of Eq. (28) is

$$\varphi = \varphi_1 - \Delta\varphi\,\mathrm{sech}^2\left(\sqrt{\Delta\varphi\tan\varphi_1}x/\Lambda\right)\,. \tag{29}$$

Velocity of the soliton in this approximation is

$$\overline{U}^2_{sol}(\varphi_1,\varphi_0) = \cos\varphi_1 - \frac{\sin\varphi_1}{2}\Delta\varphi. \tag{30}$$

The controlled quasi-continuum approximation

Let us return to Eq. (2). Looking at Eqs. (22) and (27) we realize that in the description of the kinks and solitons with $|\varphi_1| \ll 1$, the expansion parameter is φ_1^2; thus the quasi-continuum approximation (2) can be rigorously justified. However, strictly speaking, truncation of the expansion should be performed in accordance with the truncation of the series expansion of the sine function, and Eq. (4) in the consistent approximation should be written as

$$\overline{U}^2\frac{d\varphi}{dx} = \frac{d\varphi}{dx} - \frac{1}{6}\frac{d\varphi^3}{dx} + \frac{\Lambda^2}{12}\frac{d^3\varphi}{dx^3}\,. \tag{31}$$

Equation (31) clearly shows the competition between the nonlinearity, described by the second term in the r.h.s. of the equation, and the intrinsic dispersion, caused by the discreteness of the line, described by the third term. Note that (22) is the exact solution of Eq. (31) (with $\overline{U}$ given by (23)).
Looking at Eq. (29) we realize alternatively, that the quasi-continuum approximation can be rigorously justified when it is applied to the description of the solitons with $\tan\varphi_1 \cdot (\varphi_1 - \varphi_0) \ll 1$. The latter quantity is the expansion parameter in the r.h.s. of Eq. (2) in this case. So in the consistent approximation, Eq. (2) should be written as

$$\overline{U}^2\frac{d\psi}{dx} = \cos\varphi_1\frac{d\psi}{dx} - \frac{\sin\varphi_1}{2}\frac{d\psi^2}{dx} + \cos\varphi_1\frac{\Lambda^2}{12}\frac{d^3\psi}{dx^3}\,, \tag{32}$$

where $\psi = \varphi - \varphi_1$. Note that Eq. (29) is the exact solution of Eq. (32) (with $\overline{U}$ given by (30)).
Here we would like to attract the attention of the reader to the following issue. Common wisdom says that the continuum approximation and the small amplitude approximation are independent - there could be a wave with small amplitude, which allows to expand the sine function, but which varies fast in space (wavelength comparable to lattice spacing), so the continuum limit is not justified. And there could be the opposite situation (large amplitude, long wavelength), in which the sine needs to be retained but the continuum limit is allowed.
However, for the kinks and the solitons these approximations are not independent. Parametrically, the length scale of the waves is of the order of the lattice spacing Λ, so, naively, the continuum (or even the quasi-continuum) limit is not justified. What we have shown above, is that for the waves with small amplitude $|\varphi_1|$ ($\tan\varphi_1(\varphi_1 - \varphi_0)$), the length scale is $\Lambda/|\varphi_1|$ ($\Lambda/(\tan\varphi_1(\varphi_1 - \varphi_0))$), thus justifying the quasi-continuum approximation.

The shocks

Consider JTL with the capacitor and resistor shunting the JJ and another resistor in series with the ground capacitor, shown on Fig. 1 (right). As the result, Eq. (1) changes to

$$\frac{\hbar}{2e}\frac{d\varphi_n}{dt} = \left(\frac{1}{C} + R\frac{\partial}{\partial t}\right)(q_{n+1} - 2q_n + q_{n-1})\ , \tag{33a}$$

$$\frac{dq_n}{dt} = I_c\sin\varphi_n + \frac{\hbar}{2eR_J}\frac{d\varphi_n}{dt} + C_J\frac{\hbar}{2e}\frac{d^2\varphi_n}{dt^2}\ , \tag{33b}$$

where R is the ohmic resistor in series with the ground capacitor, and C_J and R_J are the capacitor and the ohmic resistor shunting the JJ.
Considering again the running wave solutions we obtain the generalization of Eq. (5)

$$\frac{\Lambda^2}{12}\frac{d^2\sin\varphi}{dx^2} + \left(\frac{C_J}{C} + \frac{R}{R_J}\right)\overline{U}^2\Lambda^2\frac{d^2\varphi}{dx^2} + \left(\frac{R}{Z_J}\cos\varphi + \frac{Z_J}{R_J}\right)\overline{U}\Lambda\frac{d\varphi}{dx} = -\sin\varphi + \overline{U}^2\varphi + F\,, \tag{34}$$

where $Z_J \equiv \sqrt{L_J/C}$ is the characteristic impedance of the JTL, and we discarded the terms with the derivatives higher than of the forth order.
We impose the boundary conditions (8) and try to understand what part of the analysis of Section can be transferred to the present case. The results (12) are determined only by the r.h.s. of Eq. (5), so are (4), following from (12). Since the r.h.s. of Eqs. (5) and (34) are identical, these equations are valid in the present case also. In particular, we obtain

$$\overline{U}^2 = \overline{U}^2_{\text{sh}}(\varphi_1, \varphi_2)\,, \tag{35}$$

which explains the subscript we introduced in Eq. (13a).
On the other hand, the resistors, by introducing the effective "friction force", break the "energy" conservation law, which means that the stationary kinks and the solitons we considered previously are no longer possible, however weak the dissipation is. However in the lossy JTL the solutions with $|\varphi_2| \neq |\varphi_1|$ (the shocks) are possible.

The continuum approximation
Looking at Eq. (34) we understand, that when C_J and/or R are large enough, and/or R_j is small enough the first term in the l.h.s. of (34) (the second term in (2)) can be discarded, hence the continuum approximation is valid, and (34) acquires Newtonian form [45]

$$\left(\frac{C_J}{C}+\frac{R}{R_J}\right)\frac{d^2\varphi}{d\tau^2}+\left(\frac{R}{Z_J}\cos\varphi+\frac{Z_J}{R_J}\right)\frac{d\varphi}{d\tau}=-\sin\varphi+\overline{U}^2\varphi+F\,, \tag{36}$$

where we have introduced the dimensionless time $\tau = x/(\overline{U}\Lambda)$. In distinction from case of the kinks and the solitons, now the fictitious particle trajectory connects the "potential energy" maximum with the "potential energy" minimum,
The shocks in the framework of the continuum approximation were studied in our previous publication [45]. In particular, in the simple case when $C_J = 0$, $R = 0$ (when (36) is similar to equation describing the motion of a fluxon in biased long JJ [42]) and for weak shock ($\Delta\varphi\cdot\cot\varphi_1 \ll 1$), where $\Delta\varphi \equiv (\varphi_1-\varphi_2)/2$, Eq. (36) takes the form

$$\frac{Z_J}{R_J}\frac{d\psi}{d\tau}=\sin\overline{\varphi}\left(\psi^2-\Delta^2\varphi\right)\,, \tag{37}$$

where $\overline{\varphi}\equiv(\varphi_1+\varphi_0)/2$ and $\psi\equiv\varphi-\overline{\varphi}$. The solution of (37) is

$$\psi=-\Delta\varphi\tanh\left(\frac{R_J}{Z_J}\Delta\varphi\sin\overline{\varphi}\cdot\tau\right)\,. \tag{38}$$

The qualitative analysis
For qualitative analysis of Eq. (34) in the general case, it is better to present it as a system of two first order differential equations

$$\left[\frac{\cos\varphi}{12}+\left(\frac{C_J}{C}+\frac{R}{R_J}\right)\overline{U}^2\right]\Lambda\frac{d\chi}{dx}=\frac{\sin\varphi}{12}\chi^2-\left(\frac{R}{Z_J}\cos\varphi+\frac{Z_J}{R_J}\right)\overline{U}\chi-\sin\varphi+\overline{U}^2\varphi+F\,, \tag{39a}$$

$$\Lambda\frac{d\varphi}{dx}=\chi\,. \tag{39b}$$

Now, one important feature of shocks can be understood immediately. We are talking about the direction of shock propagation. Linearising Eq. (39) in the vicinity of the fixed points $(\chi,\varphi)=(0,\varphi_1)$ and $(\chi,\varphi)=(0,\varphi_2)$ we obtain

$$\Lambda\begin{pmatrix} d\chi/dx \\ d\varphi/dx \end{pmatrix}=\begin{pmatrix} M_i & K_i \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \varphi-\varphi_i \\ \chi \end{pmatrix}\,,\quad i=1,2 \tag{40}$$

where

$$K_i=\frac{\overline{U}^2-\cos\varphi_i}{\cos\varphi_i/12+(C_J/C+R/R_J)\,\overline{U}^2}\,, \tag{41}$$

and here we are not interested in M_i. From the fact that φ_1 is the unstable fixed point, and φ_2 is the stable fixed point we obtain

$$\cos\varphi_2>\overline{U}^2_{\rm sh}(\varphi_1,\varphi_2)>\cos\varphi_1\,. \tag{42}$$

The inequalities (42) allow only one direction of shock propagation - from larger $\cos\varphi$ to smaller $\cos\varphi$. Taking into account (9), we can present (42) as

$$\overline{u}^2(\varphi_2)>\overline{U}^2_{\rm sh}(\varphi_1,\varphi_2)>\overline{u}^2(\varphi_1)\,, \tag{43}$$

thus establishing the connection with the well known in the nonlinear waves theory fact: the shock velocity is lower than the sound velocity in the region behind the shock, but higher than the sound velocity in the region before the shock [1].
Let us write down inequalities (42) explicitly

$$\cos\varphi_2>\frac{\sin\varphi_1-\sin\varphi_2}{\varphi_1-\varphi_2}>\cos\varphi_1\,. \tag{44}$$

We will combine the case we studied up to now, when φ_1 was the phase before the shock and φ_2 - behind the shock, with the opposite case, which corresponds to indices 1 and 2 in (44) being interchanged. The points in the phase space of the shock boundary conditions (φ_1,φ_2), for which neither (44), nor its interchanged version are satisfied, has very simple geometric property. The point (φ_1,φ_2) belongs to that region, if the secant of the curve $\sin\varphi$ between the points φ_1 and φ_2 crosses the curve, like it is shown on Fig. 4 (below). Because $\sin\varphi$ is concave downward for $0<\varphi<\pi/2$, and concave upward for $-\pi/2<\varphi<0$, it never happens if φ_1,φ_2 have the same sign. Hence the shock can exist between

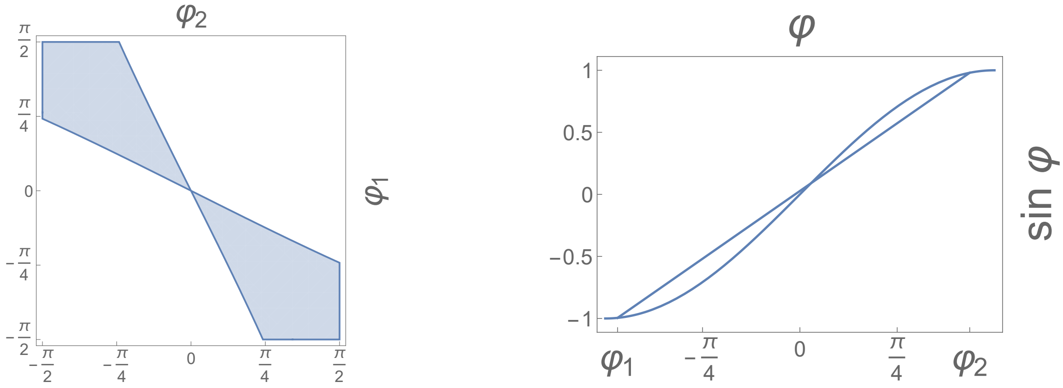


Figure 4: The phase space of the boundary conditions on the ends of the JTL φ_1 and φ_2. The region, which corresponds to the forbidden shock boundary conditions, is shaded (left). The geometric property of the points belonging to the shaded region (right).

any such points. It is not so for φ_1 and φ_2 having opposite signs. We present the phase space of shock boundary conditions on Fig. 4 (above). The forbidden region is shaded.

When the asymptotic phases on the two sides of the JTL belong to the shaded region, probably there exists some intermediate φ_{in} in between, such that the shocks between φ_1 and φ_{in}, and between φ_2 and φ_{in} are allowed. Say, when the phases are φ_1 and $-\varphi_1$, the system can chose the intermediate value $\varphi_{in} = 0$. In this hypothetical case, the shocks move in the opposite directions, and the central part with the phase $\varphi_{in} = 0$ expands with the velocity $2U_{\rm sh}(\varphi_1, 0)$. However, the case of multiple shocks being simultaneously present in the system demands further studies.

The numerical integration

Equation (34) can be easily integrated numerically. For aesthetical reasons let us simplify it by putting $R = 0$ and $C_J = 0$. (Actually, the physical meaning and the relevance of the resistor in series with the ground capacitor is not obvious. We included it because we were able to do it for free. The capacitance of the JJ is certainly physically relevant. Anyhow, when $C_J/C \ll 1$, it can be ignored.) After the simplification and substitution of the results for $\overline{U}$ and F from (4), the equation becomes

$$\frac{\cos\varphi}{12}\Lambda^2\frac{d^2\varphi}{dx^2} = \frac{\sin\varphi}{12}\left(\frac{d\varphi}{dx}\right)^2 - \frac{Z_J}{R_J}\overline{U}\Lambda\frac{d\varphi}{dx} - \frac{(\sin\varphi - \varphi_2)(\varphi_1 - \varphi) - (\sin\varphi_1 - \sin\varphi)(\varphi - \varphi_2)}{\varphi_1 - \varphi_2}\,.$$

The result of the numerical integration are shown on Fig. 5 (compare with Figs. 2 (below) and **??**).

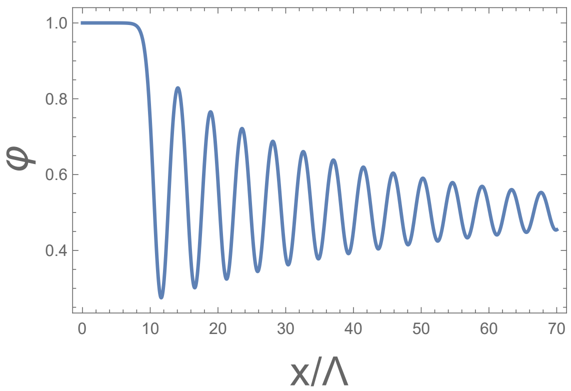


Figure 5: The shock profile according to Eq. (45). We have chosen $\varphi_1 = 1$, $\varphi_2 = .5$, $Z_J/R_J = .005$.

Dissipation is always present in real experiments. And yet we can observe solitary waves (though they are nonstationary, but practically identical to the corresponding stationary solitons at any given moment of time) in case if dissipation is weak enough. Looking at Fig. 5 we realize that weak dissipation does not completely kill solitary waves, it just makes them nonstationary/attenuating. Such solitary waves are observed in numerical calculations and in experiments, as was the case with granular chains [41, 43]. On the other hand, there is a critical rate of dissipation which transforms oscillating stationary shock waves into monotonous [47].

Discussion

Recently, quantum mechanical description of JTL in general and parametric amplification in such lines in particular started to be developed, based on quantisation techniques in terms of discrete mode operators [48], continuous mode operators [49], a Hamiltonian approach in the Heisenberg and interaction pictures [50], the quantum Langevin method [51], or on partitions a quantum device into compact lumped or quasi-distributed cells [52]. It would be interesting to understand in what way the results of the present paper are changed by quantum mechanics. Particularly interesting looks studying of quantum ripples over a semi-classical shock [53] and fate of quantum shock waves at late times [54]. Closely connected problem of classical and quantum dispersion-free coherent propagation in waveguides and optical fibers was studied recently in Ref. [55].

Finally, we would like to express our hope that the results obtained in the paper are applicable to kinetic inductance based traveling wave parametric amplifiers based on a coplanar waveguide architecture. Onset of shock-waves in such amplifiers is an undesirable phenomenon. Therefore, shock waves in various JTL should be further studied, which was one of motivations of the present work.

Conclusions

We analytically studied the running waves propagation in the discrete Josephson transmission lines (JTL), constructed from Josephson junctions (JJ) and capacitors. Due to the competition between the intrinsic dispersion in the discrete JTL and the nonlinearity, in the dissipationless JTL there exist running waves in the form of supersonic kinks and solitons. The velocities and the profiles of the kinks and the solitons were found. We also studied the effect of dissipation in the system and find that in the presence of the resistors, shunting the JJ and/or in series with the ground capacitors, the only possible stationary running waves are the shock waves, whose velocities and the profiles were also found. We have proposed the integral approximation, which is alternative to the quasi-continuum approximation.

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