Estimating the fractality of a basins of attraction using basin entropy method and sample based approach.

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Summary. Two sampled based methods, i.e. the basin stability and basin entropy used to describe the dynamics of multistable systems will be presented. Both methods are based on integrating the system's equations of motion for a large set of different initial conditions and classifying the solutions based on final attractors. The main difference between the two approaches is the different sample methods of initial conditions for each trial. In the first one, we use the random initial conditions. In contrast, for the second one, the phase space should be uniformly divided into boxes of equal size. We show under which conditions it is possible to calculate the basin entropy using random samples of initial conditions. Moreover, the basin entropy method assumes the identical size of the investigated box in all directions of multi-dimensional phase space. In many real-life systems, it is impossible to achieve; hence we introduce the scaling of the box to overcome this problem. To summarize, we show under which conditions we can accomplish the reliable value of basin entropy using random initial conditions and rescaled size of the box.

Introduction

Non-linear ODEs describing dynamical systems can be solved analytically, but it often requires simplifying equations or imposing strong assumptions on the solution. Hence, nowadays, most non-linear ODEs are tackled with numerical methods. The are several approaches that allow to investigate of the multi-stability of the systems. The problem appears for higher dimensional systems, where we can present only two-dimensional cross-sections of multidimensional phase space. Again, it is possible to overcome that, with a method proposed by Menck et al. [1] called basin stability. It can be used to characterize the volume of basins of attraction in the multidimensional phase space. To estimate the basin stability measure, one has to perform a large number of Bernoulli trials each time, drawing initial conditions randomly and checking which attractor is reached. This method is rather new, but was already successfully applied in a numerous different scenarios. The main advantages of this method are that it can be applied to all types of systems, and it is a straightforward procedure, thus a person who is not an expert on non-linear dynamics can use it to estimate the risk of unwanted behaviour. In 2017, the experimental validation of the basin stability approach has been performed [2] and results proved that the accuracy of basin stability approach is comparable with classical methods.

The disadvantage of the basin stability approach, is that it does not take into consideration the structure of the analysed basins. In 2016 Daza et al. [3] proposed a new method that include the information about the structure of the phase space. It is called a basin entropy measure and it provides information about the unpredictability of the dynamical system. To obtain basin entropy, one has to build a grid on the phase space and, in each part, estimate basin stability. Then, this value is used to obtain Gibbs entropy for every part of the grid. Summing the entropies, leads to a quantitative measure of the uncertainty associated with the state space.

The motivation is to combine these two metrics for an analysis of a dynamical system. The aim is to show, that it is possible to estimate the basin entropy accurately without distinct simulation for every part of the grid. To prove that, we will show that the basin entropy calculated using the data obtained during the estimation of basin stability is a good approximation of the one calculated classically.

Basin stability

Basin stability is defined for a n-dimensional dynamical system with N attractors in an analysed region of the state space $\Omega \subset R^n$. Then, integrating the system equation of the motion multiple times with random initial conditions from Ω allows to estimate the probability of reaching each attractor. The proportion of initial conditions that reach certain attractor to the overall number of trials is the estimation how the attractor is stable, and called basin stability $B_s(A)$ of attractor A. The application of the method is presented on an archetypal model of externally excited oscillator, the Van der Pol-Duffing system:

$$\ddot{x} - \alpha(1 - x^2)\dot{x} + x^3 = Fsin(\omega t),$$

where α, F and ω are positive constants.

Basin entropy

Basin entropy is defined for a n-dimensional dynamical system with N attractors in an analysed region of the state space $\Omega \subset R^n$. We then cover Ω with $k \in N$ disjoint n-dimensional hypercubes of linear size ε in each dimension. Each of these boxes, in principle contains infinitely many trajectories. Moreover, each of such trajectory lead to one of N attractors mentioned before. For such a formulation, the basin entropy is given by

$$S_b = \frac{1}{N}\sum_{i=1}^{k} S_i = \frac{1}{N}\sum_{i=1}^{k}\sum_{j=1}^{N_A} p_{i,j}\, log\left(\frac{1}{p_{i,j}}\right)$$

Where $p_{i,j}$is the basin stability of a j-th attractor numbered $j \in \{1, \dots N_A\}$ calculated in i-th box. The application of the method is again presented on the Van der Pol-Duffing system. Furthermore, it is shown that the value of basin entropy is not affected by scaling the state space. This is helpful in the cases, where the ranges of initial values for some state variables differ significantly from the other ones.

Estimating basin entropy with sample based methods

Basin entropy requires the initial conditions to be distributed over the state space, so that every box created during the calculation of basin entropy contains the same amount of trials. On the other hand, the basin stability initializes all trials at random, thus it cannot be ensured that the basin entropy calculated on such data matches the one equally distributed. The analysis of the relative error of basin entropy for a single multidimensional box was performed, followed by the comparison of basin entropy calculated with three different types of input data: two types of equal distribution of points per box (25 points and 100 points) and random sampling using 4,000,000 trials. It is summarized that one can accurately estimate the value of basin entropy using the random sampling method.

Trial based basin entropy for a double pendulum model

The idea behind basin stability type metrics, is to evaluate the asymptotic behaviour of a general system, where one cannot in principle visualize the results. Hence, in this section we present previously discussed metrics for a double pendulum model which is a paradigmatic example in nonlinear dynamics. We based on the real experimental rig which has been designed, constructed and tested in our laboratory. This rig was also used to experimentally validate the basin stability approach [2]. The physical model of the system is shown in Figure 1

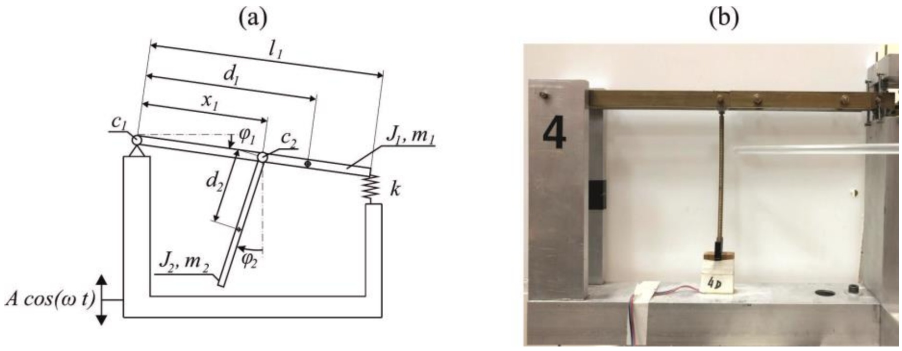

Figure 1: The physical model of the double pendulum system and its realization in laboratory.

Conclusions

The basin stability and the modified basin entropy was then used to analyse the double pendulum system. Due to the described modifications, we were able to calculate the basin entropy on the scaled state space, using the randomized trials obtained with the procedure of calculating basin stability. To choose the proper number of simulations, we performed the analysis of entropy for randomized subsets of the state space. The values of basin stability for the main attractors of the system were also presented. We detected two most stable periodic attractors, both with period 1. Finally, we calculated the basin entropy, concluding that the analysed basins are neither fractal, nor regular.

References

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