Asymptotic solutions of singular perturbed system of transport equations with small mutual diffusion in the case of many spatial variables

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Summary. We construct an asymptotic expansion on a small parameter of the solution of the Cauchy problem for a singularly perturbed system of transport equations with small nonlinearity and mutual diffusion describing the transport in a multiphase medium for many spatial variables. The asymptotic expansion of the solution is constructed as a series in powers of a small parameter and contains a functions of the boundary and inner layers. The main part of the asymptotics is described by one equation, which under certain requirements on the nonlinearity and diffusion terms is a generalization of the equation Burgers -Korteweg-de Vries in the case of many spatial variables.

Statement of the problem

The asymptotic expansion of the solution of the Cauchy problem for a singularly perturbed system of transport equations with small nonlinearity and diffusion is constructed

$$\begin{aligned} &\varepsilon^2 (U_t + \sum_{i=1}^{m} D_i U_{x_i}) = AU + \varepsilon F(U) + \varepsilon^3 \sum_{i=1}^{m} B_{ij}(U) U_{x_i x_j}, |\bar{x}| < \infty, t > 0, \\ &U(\bar{x},0) = H\omega\left(\frac{\bar{x}}{\varepsilon}\right). \end{aligned} \quad (1)$$

Here $U=\{u_1,...,u_n\}$ is the solution, $0<\varepsilon<<1$ is a small positive parameter, D_i is a diagonal constant matrix, the function $F(U)$ and the matrix $B_{ij}(U)$ are smooth enough, smooth function $\omega(x)$ is rapidly decreasing together with all derivatives. Matrix A has a single zero eigenvalue, which corresponds to the eigenvector h_0 , vector h^*_0 - eigenvector of the matrix A^T, corresponding to the zero eigenvalue, non-zero eigenvalues of the matrix A is imposed condition $Re\ \lambda <0$. Additionally, it is required that $(F(Z),h^*_0)=0$, $(B_{ij}(Z))^T h^*_0=0\ \forall Z, i,j=1,...,m$. Such systems of equations can describe the transfer of substances in multiphase media.

Construction of asymptotic expansion of the solution

The asymptotic of the solution is constructed by the method of boundary functions [3] and has the form

$$\begin{aligned} &U(\bar{x},t) = \sum_{i=0}^{N} \varepsilon^i (s_i(\bar{\zeta},t) + \pi_i(\bar{\xi},\tau)) + R_N = U_N + R_N, \\ &\bar{\zeta} = (\bar{x} - \bar{V}t)/\varepsilon, \bar{\xi} = \bar{x}/\varepsilon, \tau = t/\varepsilon^2, \bar{V} = \{(D^i h_0, h_0^*)/(h_0, h_0^*), i = 1,...,m\}. \end{aligned} \quad (2)$$

Here, the functions $s(\zeta,t)$ are functions of the inner layer (i.e., they decrease rapidly with the growth of $|\zeta|$), the functions $\pi(\xi,\tau)$ are functions of the boundary layer, i.e. they decrease rapidly with the growth of the variable τ. The number of terms of the expansion N is determined by the smoothness of the input data.
For all members of the asymptotic the equations are written out and the initial conditions are obtained.
Estimates are proved for π_i

$$\|\pi_i\| \le Ce^{-\kappa\tau}, C \ge 0, \kappa > 0. \quad (3)$$

The main term of the asymptotic is $s_0=\varphi_0(\zeta,t)\ h_0$, where φ_0 is the solution of the equation

$$\varphi_{0,t} + \sum_{i,j=1}^{m} M_{i,j}\varphi_{0,\zeta_i\zeta_j} + \sum_{i=1}^{m} (F_{eff,i}(\varphi_0))_{\zeta_i} + \sum_{i,j,k=1}^{m} (B_{eff,ij}(\varphi_0)\varphi_{0,\zeta_i\zeta_j})_{\zeta_k} = 0. \quad (4)$$

Constants M and functions $F_{eff,i}(z)$, $B_{eff,ij}(z)$ are expressed through the data of the original problem. In particular, for the quadratic function $F(U)$ and the constant matrices $B_{ij}(U)$, the equation becomes a generalization of the Burgers-Korteweg-de Vries equation to the case of many spatial variables.

Evaluation of the residual member

The residual term is estimated by the residual term in the problem.

Conclusion

1.Taking into account the estimate (3) for all $t \geq t_0 > 0$, where t_0 is any positive number independent of ε, the solution of the problem (1) is represented as

$$U(\bar{x},t) = s_0(\bar{\zeta},t) + O(\varepsilon) = \varphi_0(\bar{\zeta},t)h_0 + O(\varepsilon), \qquad (5)$$

where φ_0 is the solution of the Cauchy problem for equation (4).

2. The obtained result (5) allows us to identify non-obvious patterns of behavior of the solution of the Cauchy problem for singularly perturbed systems of type (1), as well as to identify non-obvious patterns of transfer processes in multiphase media in the case of rapid exchange between phases.

3. The numerical solution of the Cauchy problem for equation (4) requires significantly less computational resources than the solution of the original problem (1), due to the fact that equation (4) is not singularly perturbed.

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